Structural Analysis (151A)

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Time & Location:
Classes: Tue-Thur 8:00am-9:20 am, DBH 1600.
Head TA Review Sessions: Mon 2:00 pm-2:50 pm in PCB 1300.
Thu 1:00-1:50 pm in SSTR 101.
Thu 2:00-2:50 pm in SSTR 101.
Instructor office hours Tue/Thur 4:00-5:00 pm in EG 4th floor CEE Headquarther.
Head TA office hours: Thu 3:30-5:30 pm in AIRB1010.

Required Textbook:

Taking Lecture Notes:
151A lectures are based on lecture notes developed by the instructor and are substantially different from the required textbook. These notes are available on the class website under 151A section. You could also print lecture notes and work/modify them during the lecture.

Attendance Policy:
We highly encourage you to attend all classes and not even miss one. As you will gradually notice, there will be a significant difference between your lecture notes and the reference textbook. To encourage you to attend classes, some of questions in quizzes and the final exam will be the same as those solved in the class.

Feedback Policy:
In 151A, we are committed to provide you with the best teaching experience. To achieve this goal, we need your feedback to monitor our performance and your learning pace. In addition to online feedback, we welcome your constructive suggestions for improving the quality of the class.

Laptop and Cellphone Policy:
While laptops and cellphones are indispensable parts of our daily lives, they can be disruptive and distracting during a mathematically involved and theoretically challenging learning session. Therefore, laptops are not allowed in our class environment unless
otherwise stated. Also, please have your cellphones silent and in your pockets during the entire lecture.

**Course Learning Objectives:**

1. Recognize different structural systems and loads.
2. Determine stability and determinacy of various structural systems.
3. Apply methods of joints and sections to analyze statically determinate trusses.
4. Construct shear and moment diagrams of statically determinate beam systems.
5. Apply elastic beam theory, moment-area and conjugate-beam methods to calculate the elastic deformation of different structural systems under loading.
6. Apply energy methods including principles of work and energy, virtual work and Castigliano’s theorem to calculate the deflection in structural systems.
7. Calculate the influence line for beam-assembly systems.
8. Demonstrate your structural analysis knowledge by designing/building/testing a spaghetti bridge.
9. Use matrix methods and computer programs to analyze truss systems.

**Lecture Topics:**

**Week 1:**
- Lecture 0: Intro to class, spaghetti project, Type of structures and loads
- Lecture 1: determinacy and stability in beams + frames

**Week 2:**
- Lecture 2: Truss types, determinacy and stability in trusses and analysis methods
- Lecture 3: Shear and moment diagram in determined beams

**Week 3:**
- Lecture 4: Shear and moment diagram in frames regardless of their determinacy
- Lecture 5: Quiz 1, Shear/moment diagram, Elastic Beam Theory, Double Integration

**Week 4:**
- Lecture 6: Moment-area theorems and their application in beams and frames
- Lecture 7: Moment-area method for frames, Bress law and undetermined structures

**Week 5:**
- Lecture 8: Conjugate-beam method for determined and undetermined beams
- Lecture 9: external work and strain energy

**Week 6:**
- Lecture 10: Quiz 2, Principle of virtual work and the unit load method (trusses)
- Lecture 11: Application of the unit load method to 2D and 3D beams

**Week 7:**
- Lecture 12: Minimum potential energy and the first Castigliano’s theorem (trusses)
- Lecture 13: Application of the first Castigliano’s theorem to beams

**Week 8:**
- Lecture 14: Application of the first Castigliano’s theorem to Frames
- Lecture 15: Quiz 3, Influence line analysis

**Week 9**
- Lecture 16: Influence line for determined beam systems and trusses
- Lecture 17: Influence line for floor girders, moving load systems and internal loads
Week 10
Lecture 18: Matrix Analysis of truss structures
Lecture 19: Assembling global stiffness matrix and solving for unknowns

Spaghetti Bridge Contest: Week 9, Wednesday, Thursday and Friday 9am-5pm in SETH lab. We will send you details as we get close to the date.

Homework:
It is mandatory for you to form study groups to solve your weekly 151A’s homework. Please form your groups ASAP during the first week of the classes and no group changes are allowed during the quarter. You are highly encouraged to discuss within your group, however, every single student MUST individually submit a homework solution in paper. No electronic submission is allowed. The assignments are already uploaded on the class website and will be due the following Monday 9:00 am in the class drop box in front of CEE headquarter EG 4th floor. Late homework will not be graded.

Quizzes and the Final Exam:
There are three short quizzes and a final exam in this course. Scopes of these exams are described as follows:
Quiz 1 covers stability, determinacy, truss analysis structures and shear/moment diagrams.
Quiz 2 covers shear/moment diagrams and deflection calculations using elastic-beam theory, moment-area and conjugate-beam methods.
Quiz 3 covers energy methods including principle of external work and energy, virtual work principle, unit load method and Castigliano theorem.
Final exam covers the entire syllabus of the course covered in lectures 1 to 17.

These exams are all closed-book, closed-note and closed-discussion. The short quizzes will be focused on problems solved in the class, homework, examples and problem sets in the reference book. Therefore, there should be no surprises if you have studied them all carefully. The Final Exam will feature new problems that you will not find in conventional structural analysis textbooks but you will be able to solve them with techniques you learned in this course.

Spaghetti Bridge Project:
You will demonstrate your knowledge of structural analysis in this class by designing, constructing and testing a spaghetti bridge. The details of this class-wide competition are going to be discussed in details in lectures 0.

Grading Policy:
Homework: 8 p-sets (10 points).
Quizzes: 3 short closed book Quizzes (30 points: 7+8+15).
Mandatory Spaghetti Bridge Project: see below (20 points).
Final Exam: Comprehensive closed book exam (40 points).
Total points: X out of 100
Objectives of Lecture 0: (Reading Assignment: pages 3-28, 35-68)

1- Introduction to the Class Structure and Expectations
2- Spaghetti bridge contest

Objectives of Lecture 1: (Reading Assignment: pages 3-28, 35-68)

1- Types of Structures (beams, trusses, Frame, Cable, Arch)
2- Types of loads (Dead Load, Live Load, Wind Load, Earthquake)
3- Define Conventions for Loads, Supports, Structural Members
4- Define Stability and Its Necessary Conditions
5- Stability and Determinacy of Beams and Beam Assemblages
6- Stability and Determinacy of Frames + Pinned Connections

Objectives of Lecture 2: (Reading Assignment: pages 46-68, 83-136)

1- Different types of trusses
2- Stability and Determinacy of trusses
3- The principle of superposition
4- Zero force members in trusses
5- Method of joints
6- Method of sections

Objectives of Lecture 3: (Reading Assignment: pages 139-160)

1- Shear and moment diagram in determined beams
2- Determination of the approximate shape deformation in beams

Objectives of Lecture 4: (Reading Assignment: pages 160-173)

1- Shear and moment diagram in determined frames
2- Superposition of shear and moment diagrams
3- Shear and moment diagram in indeterminate beams and Frames
4- Shear and moment diagram in beam with rotational springs (covered in Pset)

Objectives of Lectures 5: (Reading Assignment: pages 305-320)

1- Quiz on the contents of Lecture 1 through 4.
2- Review of elastic beam theory from mechanics of materials
3- Double integration method

Objectives of Lectures 6: (Reading Assignment: pages 320-330)

1- First and second moment-area theorems
2- Application of moment-area method in beams
Objectives of Lectures 7: *(Reading Assignment: pages 320-330)*

1- Application of moment-area method in frames
2- Bress law for calculation of deflections in beam assemblages and frames
3- Analysis of indetermined beams using moment-area method

Objectives of Lectures 8: *(Reading Assignment: pages 330-338)*

1- Conjugate-beam method
2- Conjugate-beam method for analysis of determined beams
3- Conjugate-beam method for analysis of undetermined beams

Objectives of Lecture 9: *(Reading Assignment: pages 349-354)*

1- External work, strain energy, conservation of energy
2- Conservation of energy applied to beams

Objectives of Lecture 10: *(Reading Assignment: pages 354-363)*

1- Quiz on the contents of Lecture 3 through 8.
2- Principle of virtual work
3- Unit load method for trusses (temperature and imperfections)
4- Application of unit load method for analysis of trusses

Objectives of Lecture 11: *(Reading Assignment: pages 370-381)*

1- Unit load method for beams (temperature and settlement)
2- Analysis of 2D beams and frames with unit load method
3- Analysis of 3D beams with unit load method

Objectives of Lecture 12: *(Reading Assignment: pages 363-370)*

1- Principle of minimum potential energy and the first Castigliano’s theorem
2- Castigliano’s theorem applied to indeterminate trusses

Objectives of Lecture 13: *(Reading Assignment: pages 381-388)*

1- Formulation of the first Castigliano’s theorem for beams and frames
2- Castigliano’s theorem applied to 2D beams and frames with nodal point loads

Objectives of Lecture 14: *(Reading Assignment: pages 387-393)*

1- Castigliano’s theorem applied to 3D beams and frames with nodal point loads
2- Castigliano’s theorem applied to 3D beams and frames with distributed loads
3- Symmetric structures + symmetric and anti-symmetric loading
4- Decomposition of a general load to symmetric and anti-symmetric contributions
5- Applying ideas of parallel and series springs to analyze indeterminate structures

Objectives of Lecture 15: *(Reading Assignment: pages 539-559)*

1- Quiz on the contents of Lecture 9 through 14.
2- Influence line analysis for structures

Objectives of Lecture 16: *(Reading Assignment: pages 205-2340)*

1- Definition of the influence line and its application to beams
2- Qualitative influence line
3- Influence line for floor girders

Objectives of Lecture 17: *(Reading Assignment: pages 230-255)*

1- Application of Influence line for trusses
2- Maximum influence of a moving load system
3- Influence line of moment and shear in beams

Objectives of Lecture 18: *(Reading Assignment: pages 539-559)*

1- Computer simulation of trusses: nodes, Elements, Connectivity, support and loads
2- Stiffness matrix truss element in global coordinate system
3- Assembling stiffness matrix for a truss structure

Objectives of Lecture 19: *(Reading Assignment: pages 539-559)*

1- Static condensation and solving for nodal displacements and member forces
2- Analyzing a simple truss with the open source Matlab code
Lecture 1: Stability of beams and frames:

1. Structures
   - Planar: 2D \[ \sum F_x = 0, \sum F_y = 0, \sum M_2 = 0 \]
   - Spatial: 3D \[ \begin{align*}
   \sum F_x &= 0, \\
   \sum F_y &= 0, \\
   \sum F_z &= 0, \\
   \sum M_x &= 0, \\
   \sum M_y &= 0, \\
   \sum M_z &= 0
   \end{align*} \]

2. The focus is on planar structures, which require at least 3 support reactions for their equilibrium. Example (why do we need 3 reactions):

   - A bar with axial load
   - \[ \Sigma F_x \neq 0 \]
   - \[ \Sigma M_2 \neq 0 \]
   - \[ \Sigma F_y \neq 0 \]

3. We need 3 forces or moments that are not parallel and convergent at a single point.

4. Conventions for different types of supports with number of reactions:
   - \[ \begin{align*}
   &\text{Unstable} \\
   &\text{Stable}
   \end{align*} \]
   - \[ \begin{align*}
   V = 1 \\
   V = 2 \\
   V = 3
   \end{align*} \]
The other type of instability is due to the structure itself rather than its support. This is called internal instability. The number of constraints is:

\[ c \]

(a) Shear joint (b) Moment joint (c) Axial joint (d) Moment-axial joint

\[ c = 1 : M_L = M_R = 0 \quad c = 2 : \begin{cases} M_L = 0, & M_R = \alpha \\ M_L = \beta, & M_R = 0 \end{cases} \]

\[ U < E \rightarrow \text{unstable} \quad (n < 0) \]

\[ U = E \quad \text{might be stable, might be unstable} \quad (n = 0) \]

\[ U > E \quad \text{necessary condition not sufficient} \quad (n > 0) \]

\[ 8 \text{ Determinacy degree: } n = U - E \]

\[ \text{Unknowns: } 1 \quad \text{satisfy the equation.} \]

\[ \text{This gives us extra equation to satisfy.} \]

\[ 9 \text{ For beam: } n = R_1 - (3 + c) \]

"\( R_1 \) reactions" "\( c \) constraints at all joints"

\[ 10 \text{ Examples of beams:} \]

(a) \[ n = 2 - (3 + 1) = -1 < 0 \quad \text{unstable and under-determined} \]

Counter example:

(b) \[ n = 3 - (3 + 1) = 0 \quad \text{determined but externally unstable} \]

(c) \[ n = 4 - (3 + 1) = 1 \quad \text{undetermined but} \]
1. \( n = 1 - (3+1) = 5 - (3+1) = 1 \): 
   - Stable \& determined
   - No parallel \& convergent reactions

2. \( n = 1 - (3+1) = 4 - (3+1) = 0 \): 
   - Stable \& determined

3. \( n = 5 - (3+2) = 0 \): 
   - Determined but unstable

4. One of the best ways to show that a structure is unstable is by exerting a load and showing that equilibrium and stability conditions do not hold anymore.

5. Example:
   - \( n = 1 - (C+3) = 4 - (3+1) = 0 \): determined.
   - Also externally no parallel or convergent reactions
   - Is it stable? \( \sqrt{2} \)
   - Because it is internally unstable

6. Example:
   - \( n = 0 \) but stable.
   - \( 2T \sin \alpha = P \)

7. \( n = 6 - (3+2) = 1 \): indeterminate
   - No parallel or convergent support forces
   - Internally unstable

8. Frames are assemblies of beams such that the angles between columns and beams do not change. There is a method to calculate the determinacy of frames.
16 A: \[ n = 3 \times m + r - (3j + c) \] in equilibrium.

17 \[ n = 3 \times 15 + 3 \times 3 - (3 \times 12 \frac{1}{2} + 0) = 18 \]

In determinate and stable.

18 B: Closed-loop method: \[ n = 3 \times k + r - (c^3) \]

Closed-loops:

\[ n = 3 \times 4 + 9 - (3^3) = 18 \]

In frames, the number of constraints eqs. should be determined by practicing caution:

\[ c = 1 \quad c = 2 \quad c = m - 1 \]

\[ c = 1 \quad c = 3 \quad c = 1 \quad c = 2 \]
Example:

\[ n = 3k + r - (c + s) \]

\[ = \frac{3 \times 9 + 8}{20} - (12 + 3) = 5 \]

class accomplishment: "Wrap Up the Class"
Lecture 2: "Trusses and their Stability"

1. There are 3 types of trusses:
   - Simple
   - Compound
   - Complex

2. Simple truss: Starting from a triangular truss, if we add to elements connected at a new node and continue this process until the truss is made, such process yields a simple truss that is internally stable.

3. Examples of simple truss:

4. More complex examples starting from a stable foundation:

   Hint: Start from $A'AC$.

5. Compound truss: It is constructed from combinations of simple trusses.

Converting 3 trusses:
- Connected $\rightarrow a$: a shared joint & a member.
- Connected through $\rightarrow b$: connected through 3 non-parallel and non-convergent members.
- Connected $\rightarrow c$: "a shared joint, and 2 non-convergent support reactions at the joints position.

6. Example:
7. Connecting 3 trusses for forming a compound truss.

8. a: Connected through 3 non-aligned joints.
   b: Connected through 2 members & 2 joints (real & virtual joints are non-convergent).
   c: 1 joint & three members (virtual joint).
   d: Connected through 6 members → (c ≤ u)

9. Complex trusses are neither simple nor compound.

10. The determining degree of trusses can be obtained via:
    \[ n = m + r - 2j \]
    - \( n \) = unknowns
    - \( m \) = # of equilibrium equations in
    - \( r \) = forces in all members and supports
    - \( j \) = at each joint.
11. Stability condition:

\[
\begin{align*}
\text{a: } n &< 0 \quad \text{unstable} \\
\text{b: } n &> 0 \quad \text{stable & determinate (if no geometric instability)} \\
\text{c: } n &= 0 \quad \text{stable & indeterminant (}}
\end{align*}
\]

12. As we discussed before, \( n \geq 0 \) does not necessarily imply stability.

13. There are several methods to determine the stability of trusses:

1. Investigation method: Check the type of the truss and how members or simple trusses are connected to each other. (Good for simple & compound trusses).

2. Counter Example via Leading method: Works for \( n \geq 0 \), and a specific leading can show the unstable.

3. Zero force method for determined complex trusses: Determine zero load members, assume the force of a member to be \( s \). Determine all the other internal forces. If \( s = 0 \), this means that the structure is unstable. Otherwise, it's stable.

14. Example 1:

\[
\begin{align*}
n &= 4 + (12 + 2 \times 6) - 2 \times 14 = 28 - 28 = 0 \\
\end{align*}
\]

\( n \) is determined:

Stable \{ \begin{align*}
\text{Externally: } & 4 \text{ reactions, non-parallel & non-convergent.} \\
\text{Internally: } & 2 \text{ compound trusses connected via a support and 2 members.} \\
\end{align*} \}

\[ \Sigma M_B \neq 0 \rightarrow \text{unstable.} \]

We need 1 other member.

15. Example 2:

\[
\begin{align*}
n &= 6 + (10 + 6 + 5) - 2 \times 13 = 27 - 26 = 1 \\
\end{align*}
\]

\( n \) is indeterminant.
Example:

\[ n = 6 + (14 + 7 + 5) - 2 \times 16 \]
\[ = 32 - 32 = 0 : \text{determined} \]

Example:

\[ \Sigma M_b \neq 0 \]
\[ n = +1 \]

Example:

\[ n = 8 + (9) - 6 \times 2 = 0 : \text{determined} \]

Externally stable.

Complex truss: lets use zero load method.

Can be whatever, therefore its unstable.

Example:

\[ n = 8 + 13 - 2 \times 8 = 0 \]

Externally stable.

Counterexample: internally unstable.

Analyzing Statically determined frames:

\[ \begin{cases} \text{joints method} \\ \text{method of sections} \end{cases} \]

Consider also the superposition theorem for linear elastic structures:

Solving with method of joints.
\[ n = 3 + 19 - 2 \times 11 = 22 - 22 = 0 \]

Forces in BC:

\[ F_{BC} \times 2a - \frac{P}{2} \times 2a = 0 \quad \rightarrow \quad F_{BC} = \frac{P}{2} \]
Lecture 3: shear and moment diagrams in statically determinate structures.

Our goal is to calculate and plot moment, shear, and axial diagrams. Our conventions are as follows:

- Convention: we plot positive moment diagram on the compressive thread of the beam.
- Convention: as for shear and axial diagrams, we plot them on top of the beam.

The theory behind it is based on the equilibrium of an infinitesimal beam element. Let's assume distributed forces:

- \( w(x) \): distributed vertical load
- \( n(x) \): axial load
- \( m(x) \): moment

The most general case is beam in terms of loading:

\[
\begin{cases}
\sum F_x = 0 \quad \Rightarrow \quad \Delta N + n(x) \cdot \Delta x = 0 \\
\frac{dN(x)}{dx} = -n(x) \Rightarrow N(x) = \int_A n(x) \cdot dx
\end{cases}
\]

\[
\begin{cases}
\sum F_y = 0 \quad \Rightarrow \quad W(x) \cdot \Delta x - \Delta V = 0 \\
\frac{dV(x)}{dx} = W(x) \Rightarrow V(x) = \int_A W(x) \cdot dx
\end{cases}
\]
\[ \Sigma M_x = 0: -V(x) \Delta x + \frac{\Delta V(x)}{\Delta x} \cdot \Delta x + \omega(x) \cdot \frac{\Delta x^2}{2} + m(x) \cdot \Delta x + \Delta M - M = 0. \]

\[ \Delta M = V(x) \cdot \Delta x - m(x) \cdot \Delta x. \]

\[ \frac{dM}{dx} = V(x) - m(x) \rightarrow M(x) = \int_A^x (V(x) - m(x)) \cdot dx. \]

In case of distributed moment equal to zero:

\[ M(x) = \int_A^x V(x) \cdot dx \leftrightarrow \frac{dM}{dx} = V(x). \]

3. To simply explain things:

\[ \frac{dM}{dx} = V(x) \quad \text{slope of moment diagram} = \text{shear forces} \]

\[ \Delta M = \int_V (V(x)) \cdot dx \quad \text{change in moment} = \text{area under shear diagram} \]

\[ \frac{dV(x)}{dx} = W(x) \quad \text{slope of shear diagram} \]

\[ \Delta V = \int_V W(x) \cdot dx \quad \text{change in shear forces} = \text{area under vertical load} \]

4. Let's consider the point load and moments:

\[ N \quad \left( \begin{array}{c} \Delta M + M \\ V + \Delta V \end{array} \right) \rightarrow N + \Delta N \]

\[ \Sigma F_x = \rightarrow \Delta N = -F_1 \]

\[ \Sigma F_y = \rightarrow \Delta V = -F_1 \]

\[ \Sigma P_z = \rightarrow \Delta M = V \cdot \Delta x - M. \]
Example: determine the degree of determinacy and support forces and plot moment and shear diagrams:

\[ n = 3 \times 3 + 6 - (4 \times 3 + 2) = 15 - 14 = 1 \]

A 1 degree indeterminate → this is due to \( \sum F_x = 0 \)

→ if we assume that axial forces are negligible then the structure is stable.

\[ \sum M_B = 0 \rightarrow \frac{P}{2} \times l - M_A = 0 \rightarrow M_A = \frac{P \cdot l}{2} \]

\[ M_A \rightarrow C(x) = -\frac{P \cdot l}{2} + \frac{P \cdot x}{2} \]

\[ M_C \rightarrow E(x) = \frac{P \cdot l}{2} + \frac{P \cdot x}{2} - \frac{P \cdot (x - \frac{3l}{2})^2}{2} \]

Example:

\[ W_{\text{tot}} = \frac{W \cdot l}{2} \text{, location } \frac{2l}{3} \text{ from A, } \frac{l}{3} \text{ from B} \]

\[ \sum M_A = 0 \rightarrow W_{\text{tot}} \cdot \frac{2l}{3} - F_B \cdot l = 0 \]

\[ F_B = \frac{2}{3} \cdot W_{\text{tot}} = \frac{2}{3} \cdot \frac{W \cdot l}{2} = \frac{W \cdot l}{3} \]

\[ \sum F_Y = 0 \rightarrow F_A = \frac{W \cdot l}{6} \]

\[ \Delta V(x) = \int \frac{x}{-W(x)} \cdot dx = \left[ \frac{x}{-W(x)} \cdot \frac{x^2}{2} \right]_0^x \]
\[
\Delta V(x) = -\frac{w}{l} \cdot \frac{x^2}{2} \quad \rightarrow \quad V(x) = V(A) + \Delta V(A) = \frac{w}{l} \cdot \frac{x^2}{a} \cdot \left( \frac{a}{l} \right)^2 \\
= \frac{w}{a} \left( \frac{1}{3} - \left( \frac{x}{l} \right)^2 \right)
\]

\[
\Delta M(x) = \int_{A}^{x} V(x) \cdot dx = \int_{A}^{x} \left[ \frac{w}{b} - \frac{w}{a} \cdot \left( \frac{x}{l} \right)^2 \right] \cdot dx
\]

\[
= \frac{w}{6} + \frac{w}{6} \cdot \frac{x^3}{l} = \frac{w}{6} \cdot \left( \left( \frac{3}{l} \right) \cdot \left( \frac{x}{l} \right)^3 \right)
\]

\[
M(x) = M(A) + \Delta M(x) = \frac{w}{6} \cdot \left( \left( \frac{3}{l} \right) \cdot \left( \frac{x}{l} \right)^3 \right)
\]

2nd order polynomial.

\[
\frac{\partial M}{\partial x} = 0 \quad \Rightarrow \quad \frac{1}{l} \cdot \frac{3}{x^3} = 0 \\
\Rightarrow \quad \frac{x^2}{l} = \frac{3}{l} \Rightarrow \quad x = \frac{\sqrt{3}}{l} \approx 0.6l.
\]

\[
M_{\text{max}} = \frac{w}{6} \cdot \left( \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3\sqrt{3}} \right) = \frac{w}{18} \cdot \frac{\sqrt{3}}{l} \cdot \frac{\alpha}{3}
\]

\[
\rightarrow \quad M_{\text{max}} = \frac{w}{18} \sqrt{\frac{3}{\alpha}}
\]
Example:

\[ \begin{align*}
\sum M_D &= \sum F_E x \ell = 0 \\
\sum M_D &= P x - \frac{8x}{2} + F_A x 2\ell + F_B x \ell - P\ell_2 = 0 \\
\sum F_A + F_B &= 2P \\
\sum F_Y &= 0 \\
F_A + F_B + P &= P \\
\Rightarrow & \quad F_A = P \\
\Rightarrow & \quad F_B = 0
\end{align*} \]

\[ \begin{align*}
M_A \rightarrow B &= M_A^0 + \int_P^P \left( p - \frac{p x^2}{2\ell} \right) \, dx \\
= P x - \frac{p x^2}{2\ell} = P x \left( 1 - \frac{x}{2\ell} \right) \\
M_B \rightarrow C &= M_B^0 = PE_\ell \\
M_C \rightarrow D &= PE_\ell - P \left( x \right) \\
M_D \rightarrow E &= M_D^0 - P x
\end{align*} \]

\[ \begin{align*}
V_A \rightarrow B &= F_A + \int_0^P \frac{v x}{2} \, dx = P - P \frac{v x}{2} \\
V_B \rightarrow C &= \frac{d}{dx} V_B + F_B = 0 + 0 = 0 \\
V_C \rightarrow D &= V_C - P = 0 - P = -P \\
V_D \rightarrow E &= V_D = -P
\end{align*} \]

* Please note that whenever the shear forces are zero, the moment diagram is constant.*
(a) To determine the approximate shape of the beam deflection, we use smiley face technique. Place the eyes on the moment diagram and draw a smiley face between the beam and its moment diagram. Make sure that the boundary conditions are satisfied. Zero moment means zero bending, change in sign of moment means a change in curvature.

![Diagram of beam with smiley face technique applied](image)

Maximum deflection
Lecture 4: Shear and moment diagrams in frames, superposition of undeterminate structures and the approximate deflections.

1. Calculating shear and moment diagrams in frames is a bit more subtle as the shear forces might convert to axial forces and vice versa in joints. Therefore, it requires special attention.

2. Consider the following joint (joint A)

\[ \sum M_A = 0 \rightarrow M_R = M_L \]

\[ \begin{align*}
\sum F_x &= 0 : N_R = -N_L \cdot \cos \beta - V_L \sin \beta \\
\sum F_y &= 0 : V_R = V_L \cdot \cos \beta - N_L \cdot \sin \beta
\end{align*} \]

3. Let's consider a special case of \( \beta = 90^\circ \).

\[ \begin{align*}
M_R &= M_L \\
N_R &= V_L \\
V_R &= -N_L
\end{align*} \]

You see, the shear and axial forces change place at the joint location.
Example: determine shear and moment diagrams:

\[ \sum F_x = 0 \rightarrow N_A = P \]
\[ \sum M_z = 0 \rightarrow M_A = \]
\[ \sum F_y = 0 \rightarrow \]

\[ M_{AB} = P \ell \]
\[ A: \] \[ v_{AB} = P \]
\[ N_{AB} = 0 \]

\[ M_{BC} = P \ell \]
\[ B: \] \[ v_{BC} = 0 \]
\[ N_{BC} = P \]

\[ M_{CD} = \frac{P \ell}{2} \cdot P \ell \]
\[ C: \] \[ v_{CD} = \pm -P \]
\[ N_{CD} = 0 \]

Please note that points A and C have moved to the left.

Please note that the joints remain rigid, i.e., the angles do not change.
Example 2: determine shear and moment diagram:

\[
\sum F_x = 0 \rightarrow N_A = P
\]

\[
\sum M_A = 0 \rightarrow F_D = P
\]

\[
\sum M_D = 0 \rightarrow F_A = P
\]

Note the direction.

AB:

\[
M_{AB} = P \ell
\]

\[
V_{AB} = P
\]

\[
N_{AB} = P
\]

BC:

\[
M_{BC} = PL - P \ell
\]

\[
V_{BC} = -P
\]

\[
N_{BC} = 0
\]

V:

\[
-P
\]

M:

\[
PC
\]

Strength line.
Example 3: Inclined Loading on a Frame.

\[ \sum P_x = 0 \rightarrow N_A = w \cdot l \]

\[ \sum F_y = 0 \rightarrow F_c = F_A \]

\[ \sum M_A = 0 \rightarrow F_c \cdot l - w e \cdot l/2 = 0 \rightarrow F_c = \frac{w e}{4} \]

\[ \sum M_x = 0 \rightarrow M + w \cdot \frac{x \sqrt{2}}{2} + \frac{w e}{4} \cdot \frac{x \sqrt{2}}{2} - w e \cdot \frac{x \sqrt{2}}{2} = 0 \]

\[ M = w e \cdot \frac{\sqrt{2}}{2} - w e \cdot \frac{x \sqrt{2}}{2} = \frac{3 w e}{8} \cdot \frac{\sqrt{2}}{2} - \frac{w e \cdot \sqrt{2}}{2} \]

\[ M_{AB} = \frac{w e (3 \sqrt{2} - x)}{4} \]

\[ \sum F_x = 0 \rightarrow V + w \frac{\sqrt{2}}{2} \cdot x - w e \cdot \frac{\sqrt{2}}{2} + \frac{w e}{4} \cdot \frac{\sqrt{2}}{2} = 0 \]

\[ \rightarrow V_{AB} = w e \frac{\sqrt{2}}{4} - w e \frac{\sqrt{2}}{2} = \frac{3 w e}{8} \]

\[ \sum M_x = 0 \rightarrow M + \frac{w e}{2} \cdot x \cdot l - w e \cdot x \cdot l + w e \cdot l/2 = 0 \]

\[ \rightarrow M = 0 \]

\[ \sum P_y = 0 \rightarrow V = - \frac{w e}{4} \]
Similar to the superposition of loads in structures to calculate reaction, we can superpose internal moments and shear diagrams.

Example: use superposition technique to plot moment diagrams.

For the shear & moment diagram in indeterminate structures:

\{ A: \text{determine the degree of indeterminacy: } g \\
B: \text{assume the } n \text{ reactions or internal forces as known} \\
C: \text{calculate the rest of reactions and internal forces with respect to those values} \}
Example: Determine the shear and moment diagram in the structure below.

\[ n = 3 + 4 - 2 \times 3 = 1 \]

Assume force in B = \( P_B \)

\[
\begin{align*}
\sum F_y &= 0 \quad \rightarrow \quad F_A = P - F_B \\
\sum F_x &= 0 \quad \rightarrow \quad N_A = 0 \\
\sum M_A &= 0 \quad \rightarrow \quad M_A = P \cdot l - F_B \cdot 2l
\end{align*}
\]

**AC**:

\[
\begin{align*}
N_{AC} &= 0 \\
V_{AC} &= P - F_B \\
M_{AC} &= P \cdot l - 2F_B \cdot l - P_x + F_B \cdot x
\end{align*}
\]

**CB**:

\[
\begin{align*}
N_{CB} &= \cdot \\
V_{CB} &= -F_B \\
M_{CB} &= -P_B \cdot l + F_B \cdot x
\end{align*}
\]

Location of \( S_{max} \) unknown.
Lecture 5: Elastic Beam Theory and Double Integration Method

1. Consider the following beam, before and after loading:

\[ \frac{\Delta \text{length}}{\text{length}} = \frac{dS' - ds}{ds} \]

\[ \frac{1}{\rho} = -\frac{My}{I_{zz}} \]

\[ \frac{1}{\rho} = \frac{d^2\nu}{dx^2} \quad ; \quad \theta \ll 1 \rightarrow \theta \approx 0 \]

\[ \frac{d^2\nu}{dx^2} = \frac{M(x)}{EI} \]

\[ \sigma_{xx} = \frac{E}{y} \]

\[ \sigma_{xx} = \frac{M(y)}{I_{zz}} \]

For linear, elastic, homogeneous beam:

\[ \epsilon_{yy}(y) = \frac{1}{E} \sigma_{xx}(y) \]

Infinitesimal beam section.

 Neutral axis.

\[ \text{extension of } ds \]

\[ \text{lateral deflection} \]

\[ u \]

\[ \text{second order ordinary differential equation} \]
(2) Steps to calculate the deflection:

A: Calculate support forces and moments;

B: Plot the moment diagram.

C: Use the integration method at different beam segments.

D: Determine the constants.

(3) Example: Find the maximum deflection in the beam:

\[ M_{A} = \frac{Pbx}{a+b}, \quad 0 < x < a \]

\[ M_{B} = \frac{Pa(l-x)}{a+b}, \quad a < x < b \]

\[ a+b = l. \]

\[ V_1' = \int \int \frac{M}{EI} \cdot dx^2 = \int \frac{Pbx}{a+b} \cdot dx = \frac{Pb^3x^3}{6EI} + C_1x + C_2 \]

Since \( V_1(0) = 0 \) \( \Rightarrow C_2 = 0 \) and \( \theta_A = C_1 \)

\[ V_2' = \int \int \frac{M}{EI} \cdot dx^2 = \int \frac{Pa(l-x)}{a+b} \cdot dx = \frac{Pa(l-x)^3}{6EI} + C_3x + C_4 \]

Since \( \theta_2(l) = 0 \) \( \Rightarrow C_3l + C_4 = 0 \) \( \Rightarrow C_4 = -C_3l \)

\[ V_2' = \frac{Pa(l-x)^3}{6EI} - C_3(l-x) \]

\[ \theta_B = C_3 \]

Therefore:

\[ \begin{cases} V_1' = \frac{Pb^3x^3}{6EI} + \theta_A x, \\
V_2' = \frac{Pa(l-x)^3}{6EI} - \theta_B (l-x) \end{cases} \]

B.C.: \( V_1(a) = V_2(a) \)

\[ \theta_1(a) = \theta_2(a). \]
\[
\begin{align*}
\begin{cases}
\theta_A + b\theta_B &= \frac{Pab}{6EI} (b-a) \\
\theta_A - \theta_B &= -\frac{Pba}{2EI}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\theta_A &= -\frac{Pb(e^2-b^2)}{6EI} \\
\theta_B &= \frac{Pb(e^2-a^2)}{6EI}
\end{align*}
\]

\[
\begin{align*}
\mathcal{V}_1 &= \frac{Pbx^3}{6EI} - \frac{Pb(e^2-b^2)}{6EI} \\
\alpha &= \frac{Pbx}{6EI}
\end{align*}
\]

\[
\begin{align*}
\frac{\mathcal{V}_1}{\alpha} &= 0 \\
\frac{\partial}{\partial x} \left( x^3 + xe^2 + ab^2 \right) &= 0 \\
3x^2 + e^2 + b^2 &= 0 \\
\alpha_{\text{max}} &= \sqrt{\frac{e^2-b^2}{3}}
\end{align*}
\]

\[
\begin{align*}
\mathcal{V}_{\text{max}} &= \mathcal{V}_1 (x = \sqrt{\frac{e^2-b^2}{3}}) = \frac{Pb(e^2-b^2)^{3/2}}{a\sqrt{3}EI}
\end{align*}
\]

4. As we see, calculating the maximum deflection through double integration method is demanding. Is there an easier way to calculate deflection and angle change without going through a lot of problems? This is the aim of the next 12 sessions (lectures).
Lecture 6: Moment Area method and application in frames and frames.

1. \( \frac{d^2 \gamma}{dx^2} = \frac{M(x)}{EI} \rightarrow \frac{d}{dx} \left( \frac{d \gamma}{dx} \right) = \frac{M(x)}{EI} \rightarrow \frac{d \theta}{dx} = \frac{M(x)}{EI} \).

\[ \rightarrow d \theta = \frac{M(x)}{EI} \cdot dx. \]

\[ \theta_{A/B} = \int_0^\ell d \theta \quad \& \quad \delta_{A/B} = \int_0^\ell x \cdot d \theta \]

2. First moment area theorem:

\[ \theta_{A/B} = \int_0^\ell d \theta = \int_0^\ell \frac{M(x)}{EI} \cdot dx = \frac{A_H}{EI} \quad \text{area under moment diagram.} \]

Assuming EI constant.

3. Second moment area theorem:

\[ \delta_{B/B} = \int_0^\ell x \cdot d \theta = \int_0^\ell x \cdot \frac{M(x)}{EI} \cdot dx = \frac{\ell A_m}{EI} \quad \text{area under moment diagram.} \]

Assuming EI constant.
4) Example: Determine $\theta_B$ and $S_B$? (Assume $EI$ = constant).

\[ \theta_B = \frac{Ple}{2EI} = \frac{P^2}{2EI} \]

\[ S_B = S_{BA} = \frac{Ml}{EI} = \frac{Ple^2}{2EI} \times \frac{2l}{3} = \frac{Ple^2}{3EI} \]

\[ M = \frac{we}{2} - \frac{wx^2}{2} \]

\[ \theta_A = \theta_B = \theta_{A/C} = \theta_{B/c} \]

\[ \theta_A = \frac{2}{3} \cdot \frac{e}{2} \cdot \frac{we^2}{8EI} = \frac{we^3}{24EI} \]

\[ \delta_C = \frac{we}{(24EI) \times \frac{S}{12}} \cdot \frac{e}{2} = \frac{Swe^4}{384EI} \]

\[ \delta_C = \theta_A \cdot \frac{e}{2} - \delta_{BA} = \frac{we^3}{24EI} \times \frac{e}{2} - \frac{we^3}{24EI} \times \frac{3}{8} \times \frac{e}{2} \]

\[ = \frac{5we^4}{384EI} \]

5) Example: Determine $\theta_A, \theta_B, \theta_C, \delta_C, \delta_{max}$? (Assume $EI$ = constant $\times a/b$).

\[ \theta_A = \frac{S_{BA}}{l} = \frac{l}{2} \cdot \frac{Pab}{EI} \cdot \frac{e}{3} = \frac{Pab(l + b)}{6EI} \]

\[ \theta_B = \frac{Pab(l + a)}{6EI} \]

\[ \theta_A = \theta_C + \theta_{A/c} \rightarrow \theta_C = \theta_A - \theta_{A/c} \]
\[ \theta_c = \frac{P_a b (l+b)}{6 EI l} - \frac{P_a b}{2 E I l} - \frac{P_a (l+b-3a)}{6 E I} = \frac{2 P_a b (b-a)}{6 E I} = \frac{P_a b (b-a)}{3 E I} \]

\[ \left\{ \begin{array}{l} b > a \rightarrow \theta_c : \text{clockwise similar to the shape above} \\
 b = a \rightarrow \theta_c = 0 \\
 b < a \rightarrow \theta_c : \text{anti-clockwise rotation} \end{array} \right. \]

\[ \delta_c = a \cdot \theta_a - \delta_{cH} = \frac{P_a b (l+b)}{6 E I l} - \frac{P_a b}{2 E I l} \cdot \frac{a}{3} = \frac{P_a^2 b^2}{3 E I l} \]

\[ \theta_{B/H} = \theta_B \rightarrow \frac{P_a b (l+a)}{6 E I l} = \frac{x}{2} \cdot \frac{P_a x}{E I} \]

\[ x = \sqrt{\frac{b (l+a)}{3}} \]

\[ S_{\text{max}} = \theta_B \cdot x - \delta_{cB} = \sqrt{\frac{b (l+a)}{3}} \cdot \frac{P_a b (l+a)}{6 E I} - \frac{P_a b (l+a)}{6 E I} \cdot \frac{1}{3} \sqrt{\frac{b (l+a)}{3}} \]

\[ = \frac{P_a b (l+a)}{6 E I} \cdot \sqrt{\frac{b (l+a)}{3}} \]

7 Example: Determine \( \theta_B, \theta_c, \theta_c \) and \( S_{\text{max}} \)? \( (EI)_{AC} = \frac{1}{2} (EI)_{CB} \)

\[ M_0. \quad S_{B/A} = \int_A^B \frac{M x}{EI} \cdot dx \]

\[ = \frac{1}{2} \cdot \frac{P}{2} \cdot \frac{ME}{2EI} \cdot (l/2 + l/6) + \]

\[ + \frac{P/2}{4EI} \cdot l/4 + \frac{1}{2} \cdot \frac{P}{2} \cdot \frac{M}{4EI} \cdot l/6 = \frac{ME^2}{8 EI} \]
\[
\theta_A = \frac{S_{Bl/A}}{E} = \frac{ML}{8EI}
\]

\[
\theta_B = \frac{S_{Al/B}}{E} = \frac{1}{E} \left( \frac{1}{2} \cdot \frac{L}{2EI} \cdot \frac{M}{3} \cdot \theta_2 \right) + \frac{L}{2} \cdot \frac{M}{4EI} \left( \frac{3}{4} \cdot \theta_2 \right)
\]

\[
+ \frac{L}{2} \cdot \frac{M}{4EI} \left( \frac{1}{3} + \frac{1}{2} \right) = \frac{ML}{EI} \left( \frac{1}{4} + \frac{3}{9} + \frac{5}{16} \right)
\]

\[
= \frac{11}{48} \frac{ML}{EI}
\]

\[
\theta_{AC} = \theta_A + \theta_C 
\]

\[
\theta_C = \frac{M_0}{2} \cdot \frac{1}{2} \cdot \frac{L}{EI} - \frac{ML}{8EI} = 0
\]

Therefore, \( C \) is the point of maximum deflection as the slope is horizontal.

\[
\delta_C = \frac{S_{AC}}{EI} = \frac{M_0}{2} \cdot \frac{1}{2} \cdot \frac{L}{2EI} \cdot \frac{L}{3} \cdot \frac{1}{2} = \frac{M_0 L^2}{24EI}
\]

Example: Determine \( \theta_B, \theta_C, \theta_D, \delta_C, \delta_A, \delta_B \) and \( \theta^\text{\(P\)} \) in the following frame.

\[ EI = \text{const.} \]
\[ \theta_B = \theta_{A/B} = \frac{PE_h \ell}{2EI} \cdot \frac{1}{2} = \frac{PE^2}{2EI}, \quad S_{BB} = 0 \rightarrow \text{axial deformation is neglected} \]

\[ S_B = S_{B/A} = \frac{PE^2}{2EI} \cdot \ell / 3 = \frac{PE^3}{6EI} \]

\[ \theta_c = \theta_B + \theta_{B/C} = \frac{PE^2}{2EI} + \frac{PE^2}{EI} = \frac{3PE^2}{2EI}, \quad S_{B}^H = S_{C}^H \rightarrow \text{no axial deformation} \]

\[ S_c = \ell \cdot \theta_B + \delta_{c/B} = \frac{PE^3}{2EI} + \frac{PE^2 \ell}{EI} \cdot \frac{1}{2} = \frac{PE^3}{EI} \]

\[ \theta_D = \theta_C + \theta_{D/C} = \frac{3PE^2}{2EI} + \frac{PE^2}{2EI} = \frac{5PE^2}{2EI} \]

\[ S_D = S_{B/C} + \ell \theta_C = -\frac{PE^3}{6EI} + \frac{3PE^3}{2EI} + \frac{PE^3 \ell}{3EI} = \frac{5PE^3}{3} \]

\[ \delta_D = \delta_{D/C} = \frac{PE^3}{EI} \]
1. Let's continue with frame examples. Determine: $\theta_A$, $\theta_B$, $\theta_C$, $\theta_D$. $\delta_B^H = \delta_C^H = \delta_B^H = \delta_C^H$.

$$\theta_B = \frac{S_{BC}}{E} = \frac{1}{E} \left( \frac{PE^2}{2EI} \cdot \frac{2l}{3} \right) = \frac{PE^2}{3EI}$$

$$\theta_C = \frac{S_{CB}}{E} = \frac{1}{E} \left( \frac{PE^2}{2EI} \cdot \frac{l}{3} \right) = \frac{PE^2}{6EI}$$

$$\theta_A = \theta_B + \theta_{AB} = \frac{PE^2}{3EI} + \frac{PE^2}{2EI} = \frac{5}{6} \frac{PE^2}{EI}$$

$$\theta_D = \theta_C = \frac{PE^2}{6EI}$$

$$\delta_B^H = E \theta_A - \delta_{BA} = \frac{E}{6} \frac{PE^2}{EI} - \frac{PE^2}{2EI} \cdot \frac{l}{3} = \frac{2}{3} \frac{PE^2}{EI}$$

$$\delta_C^H = \delta_C^H + \theta_C \cdot \frac{2PE^2}{3EI} + \frac{PE^2}{6EI} = \frac{5}{6} \frac{PE^2}{EI}$$
Example: Determine $\theta_A$, $\theta_B^L$, $\theta_B^R$, $\theta_C$, $\theta_D =$ and $S_B^V$.

\[ \theta_C = \frac{S_{DC}}{E} = \frac{1}{E} \left( \frac{P^2 e^2}{2EI} \right) = \frac{P^2}{2EI} \]

\[ \theta_D = \frac{S_{CD}}{E} = \frac{1}{E} \left( \frac{P^2 e^2}{2EI} \cdot \frac{e}{3} \right) = \frac{P^2}{6EI} \]

\[ S_B^V = l \cdot \theta_C + S_{B\{C\}} = \frac{P^2 e^3}{3EI} + \frac{P^2}{2EI} \cdot \frac{e}{3} l = \frac{2}{3} \frac{P^2 e^3}{EI} \]

\[ \theta_B^L = \frac{S_B^L}{l} = \frac{2}{3} \frac{P^2}{EI} \]

\[ \theta_B^R = \theta_C + \theta_{BCC} = \frac{P^2}{8EI} + \frac{P^2}{2EI} = \frac{5}{6} \frac{P^2}{EI} \]
Example: Determine $\theta_A = ?$

\[\begin{align*}
M & \left\{ \begin{array}{c}
BC: M = \frac{P}{1+\sqrt{2}} \cdot x \\
AB: M = \frac{P}{1+\sqrt{2}} \cdot x
\end{array} \right.
\end{align*}\]

\[\begin{align*}
\theta_A &= \theta_B + \theta_{A/B} \\
\theta_B &= \theta_A - \frac{Pe^2}{2EI(1+\sqrt{2})}
\end{align*}\]

\[\begin{align*}
\delta_B &= l\theta_A - \delta_{B/A} = l\theta_A - \frac{Pe^3}{6EI(1+\sqrt{2})}
\end{align*}\]

\[\begin{align*}
\delta_B &= \delta_B^H = \delta_B^V = \frac{\sqrt{2}}{2} \delta_B
\end{align*}\]

\[\begin{align*}
\delta_C &= \delta_C^V + l\theta_B - \delta_{C/B} = 0
\end{align*}\]

\[\begin{align*}
\delta_C^V &= \frac{\sqrt{2}}{2} \theta_A - \frac{Pe^3}{12EI(1+\sqrt{2})} + l\theta_A - \frac{Pe^3}{2EI(1+\sqrt{2})}
\end{align*}\]

\[\begin{align*}
\delta_C^V &= \frac{\sqrt{2}}{2} \theta_A - \frac{Pe^3}{3(1+\sqrt{2})EI} = 0
\end{align*}\]

\[\begin{align*}
\theta_A &= \left(1 + \frac{\sqrt{2}}{2}\right) = \frac{Pe^3}{EI(1+\sqrt{2})} \\
\theta_A &= \left(1 + \frac{\sqrt{2}}{2}\right)
\end{align*}\]
\[ \theta_B = \theta_A + \sum \frac{A_i}{EI}. \]
\[
\begin{align*}
S_B^v &= S_A^v + \theta_A (x_B - x_A) + \sum_{i=1}^n \frac{A_i}{EI} (x_i - x_B) \\
S_B^h &= S_A^h + \theta_A (y_B - y_A) - \sum_{i=1}^n \frac{A_i}{EI} (y_i - y_B).
\end{align*}
\]

This is the so-called "Bress Law".

5. Example from last lecture:

\[ \Theta_B = -\frac{P e^2}{2EI}, \quad S_B^v = 0, \quad S_B^h = \frac{P e^2}{2EI}, \quad l_3 = \frac{P e^3}{6EI}. \]

\[ \Theta_C = -\frac{3P e^2}{2EI}, \quad S_C^v = -\frac{P e^2}{2EI} l - \frac{P e^2}{EI} l_2 = -\frac{P e^3}{EI}, \quad S_C^h = S_B^h. \]

\[ \Theta_D^h = -\frac{2P e^2}{EI}, \quad S_D^h = \frac{P e^2}{2EI} \left( -\frac{2}{3} l \right) + \frac{P e^2}{EI} (-l) + \frac{P e^2}{2EI} (-2l_3) \]
\[
= -\frac{5P e^3}{3EI}.
\]
6 Example: Determine $\theta_A$?

\[
\delta_D^H = -\theta_A (2l) + \frac{WE^2}{8EI} \left( 2 - \frac{2}{3} l \right) - \frac{WE^3}{12EI} (0) - \frac{WE^3}{8EI} \left( \frac{2l}{3} \right) = 0
\]

$\theta_A = 0$

7 Indeterminate Example: Determine the support reaction $R_B$?

\[
\delta_B^v = (\delta_B^v)' + (\delta_B^v)^2 = 0 \rightarrow \frac{5PE^2}{48EI} = \frac{R_B l^3}{3EI} \rightarrow R_B = \frac{5}{16} P
\]

$\delta_B^v = \delta_{C/A} + \theta_C l/2 = \frac{PL}{8EI} + \frac{PL}{8EI} \times \frac{l}{3} = \frac{5PL^3}{48EI}$

$\delta_B^v = \delta_{BA} = -\frac{R_B l^3}{2EI} \times \frac{2l}{3} = \frac{R_B l^3}{3EI}$
This method was initially proposed by Müller and Breslau in 1865 to calculate the displacement in beams. Consider the following infinitesimal beam element:

\[ \sum F_y = 0 \rightarrow dV = w \cdot dx. \]

\[ \sum M_0 = 0 \rightarrow V \cdot dx + M \cdot dM + w \cdot dx \cdot \frac{dx}{2} = 0 \]

Also, from elastic beam theory we know that:

\[ \frac{dV}{dx} = w. \]

\[ \frac{dM}{dx} = V \]

\[ \frac{d\theta}{dx} = \frac{M}{EI} \]

The set of (1) and (2) equations are very similar, in a sense that:

\[ w \rightarrow \frac{M}{EI} \quad V \rightarrow \theta \]

This analogy suggests that, if we apply a force to a load of \( \frac{M}{EI} \), its shear and moment diagrams are equivalent to the rotation and deflection, respectively. Special attention should be given to the supports as they have to satisfy the rotation and displacement B.C.s.
The relevant boundary conditions between the real and conjugate beam are listed below:

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<tr>
<th>Real Beam</th>
<th>Conjugate Beam</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta = \theta_1$</td>
<td>$\phi = \phi_1$</td>
</tr>
<tr>
<td>$y = 0$</td>
<td>$y = 0$</td>
</tr>
<tr>
<td>$v = v_1$</td>
<td>$v' = v'_1$</td>
</tr>
<tr>
<td>$M = M_1$</td>
<td>$M' = M'_1$</td>
</tr>
</tbody>
</table>

(1)

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</thead>
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<td>$\phi = \phi_2$</td>
</tr>
<tr>
<td>$y = y_1$</td>
<td>$y = y_1$</td>
</tr>
<tr>
<td>$v = v_2$</td>
<td>$v' = v'_2$</td>
</tr>
<tr>
<td>$M = M_2$</td>
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</table>

(2)

<table>
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<th>Conjugate Beam</th>
</tr>
</thead>
<tbody>
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<td>$\theta = \theta_3$</td>
<td>$\phi = \phi_3$</td>
</tr>
<tr>
<td>$y = 0$</td>
<td>$y = 0$</td>
</tr>
<tr>
<td>$v = v_3$</td>
<td>$v' = v'_3$</td>
</tr>
<tr>
<td>$M = M_3$</td>
<td>$M' = M'_3$</td>
</tr>
</tbody>
</table>

(3)

<table>
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<tr>
<th>Real Beam</th>
<th>Conjugate Beam</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta = \theta_4$</td>
<td>$\phi = \phi_4$</td>
</tr>
<tr>
<td>$y = y_4$</td>
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<tr>
<td>$v = v_4$</td>
<td>$v' = v'_4$</td>
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<td>$M = M_4$</td>
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(4)

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<th>Conjugate Beam</th>
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<td>$\phi = \phi_5$</td>
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(5)

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<td>$v' = v'_6$</td>
</tr>
<tr>
<td>$M = M_6$</td>
<td>$M' = M'_6$</td>
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</table>

Here is the procedure for conjugate beam analysis:

1. Plot the shear and moment diagram.
2. Plot the conjugate beam based on the above table.
3. Insert a load of $M_x$ on the conjugate beam.
4. Calculate the shear and moment diagram for the shear and moment diagram.
Example: Plot the conjugate beam in the following beams:

\[ A \rightarrow \quad B \rightarrow \quad C \rightarrow \quad D \rightarrow \]

\( E \): The shear in conjugate beam is \( \theta \) and moment is deflection.

5. Example: Plot the conjugate beam in the following beams:

\[ A \quad B \quad C \quad D \quad E \]

\[ \text{determinate.} \]

6. If a beam is indeterminate, its conjugate beam will be indeterminate unstable.

   If a beam is determinate, its conjugate beam will be determinate.

7. Example: Plot the conjugate beam of the following beam sections:

\[ A \rightarrow \quad B \rightarrow \]

\( \text{The displacement at } A \text{ is known to be equal to } -\frac{R_A}{K}. \)

\( \text{The story is the same as above.} \)
Example: Calculate the following values using conjugate beam method:

\[ \delta_c, \delta_B, \delta_D, \theta_{RB}, \text{ and } \theta_L. \]

Conjugate Beam:

\[ M = \frac{P}{E} \]

\[ \sum M = 0 \rightarrow R_B' \cdot l = \frac{P \cdot l}{2} \cdot \frac{l}{3} \rightarrow R_B' = \frac{P l^2}{6EI} \]

\[ 3 \sum M_E = 0 \rightarrow \frac{P l^2}{6EI} \cdot 3l - \frac{P l}{2EI} \cdot 2l \cdot \frac{1}{2} - R_D' \cdot l = 0 \]

\[ R_D' = \frac{3P l^2}{2EI} \]

\[ \sum F_y = 0 \rightarrow \frac{P l^2}{6EI} + \frac{3P l^2}{2EI} + R_E - \frac{P l^2}{EI} = 0 \rightarrow R = -\frac{2P l^2}{3EI} \]
\[
\theta_{LB} = \frac{V_{LB}}{EI} \quad \theta_{RB} = \frac{V_{RB}}{EI} = \frac{R_{B}'}{EI} = \frac{PL^2}{6EI}
\]

\[
\delta_B = \frac{M_B}{EI} = 0 \quad \delta_C = \frac{M_C}{EI} = \frac{PL^2}{6EI} - \frac{PL^2}{3EI} = \frac{-PL^2}{3EI}
\]

\[
\delta_D = \frac{M_D}{EI} = -\frac{2PL^2}{3EI} \quad e = -\frac{2}{3} \frac{PL^3}{EI}
\]

downward deflection.

The approximate deformation of the beam is as follows:

![Beam deformation diagram]

9) Example: Calculate the following values using conjugate beam method:

\[\sum F_y = 0 \rightarrow R_c = we \]

\[\sum M_c = 0 \rightarrow M_A = wLxL/2 = W\ell^2/2 \]

\[w^2 - wLxL = \frac{w}{2}(L \ell_x^2) \]

\[\begin{align*}
M'_{B} &= \frac{we^2}{2} \\
M'_{L} &= \frac{we^2}{2}
\end{align*} \]

conjugate beam:

![Conjugate beam diagram]

The \( M_B \) moment should be calculated using beam equilibrium equations.
\[ \sum P_y = 0 \rightarrow P_c' = \frac{w l^3}{2EI} + \frac{2}{3} x l x \frac{w l^2}{2EI} = \frac{5}{6} \frac{w l^3}{EI} \]

\[ \sum M_o = 0 \rightarrow M_b' = \frac{w l^3}{2EI} \cdot \frac{3l}{8} + \frac{2l}{3} \frac{w l^3}{EI} x \frac{5}{8} l = \frac{Q_3}{24} \frac{w l^3}{EI} \]

Now, let's calculate the deflections and rotations:

\[
\begin{align*}
\theta_R &= \nu_R' = \frac{w l^3}{2EI} \\
\theta_C &= \nu_C' = \frac{5}{6} \frac{w l^2}{EI} \\
\delta_{LB} &= M_{LB}' = \frac{w l^4}{4EI} \\
\delta_{RB} &= M_{RB}' = \frac{w l^4}{4EI} - M_1 = -\frac{17}{24} \frac{w l^4}{EI}
\end{align*}
\]
Lecture 9: Conservation of Energy, Strain energy and external work.

1. Conservation of energy principle states that the work done by all the external forces acting on a structure, $U_e$, is transformed into internal work or strain energy, $U_i$, which is developed when the structure deforms.

$$ U_e = 0 $$

2. If the structure remains elastic during the entire loading process, the structure will return to its initial undeformed structure when unloaded.

3. So, the question is how to calculate $U_e$ and $U_i$ in terms of external loading and internal moment? We elaborate on this a follows:

4. Consider a fixed bar at one side and a quasi-static load applied on the other side.

$$ S = \frac{P \cdot l}{EA} \Rightarrow \sigma = \frac{P}{A} \quad \varepsilon = \frac{S}{E} $$

strain energy will be stored in this beam very similar to a spring.

$$ \sigma_{xx} = \frac{P}{A} \quad \varepsilon_{xx} = \frac{\sigma_{xx}}{E} \frac{P}{AE} $$

$$ \sigma_{yy} = 0 \quad \varepsilon_{yy} = 0 \quad \varepsilon_{zz} = -\frac{\sigma_{zz}}{E} \frac{P}{AE} $$

$$ \sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = \sigma_{zy} = 0 $$
\[ U_i = \int \int \sigma : d\varepsilon \, dV \]

Linear elastic material:

\[ = \int \int \frac{1}{2} \left( \sigma_{xx} \varepsilon_{xx} + \sigma_{yy} \varepsilon_{yy} + \sigma_{zz} \varepsilon_{zz} + \sigma_{xy} \varepsilon_{xy} + \sigma_{xz} \varepsilon_{xz} + \sigma_{yz} \varepsilon_{yz} \right) \, dV \]

In fact, each strain mode is capable of storing energy independently of the other strains.

\[ = \frac{1}{2} \int \int \sigma_{xx} (x, y, z) \varepsilon_{xx} (x, y, z) \, dV \]

The volume of the beam.

\[ = \frac{1}{2} \frac{P^2}{A^2 E} \int \int \, dV = \frac{1}{2} \frac{P^2}{A^2 E} \times A \times l = \frac{P^2 l}{2EA} \]

Quasi-static loading.

\[ U_e = \frac{1}{2} P \times s \]

\[ U_e = U_i \rightarrow \frac{1}{2} \frac{P^2 L}{A^2 E} = \frac{1}{2} \frac{P}{A} \times s \]

\[ s = \frac{P L}{EA} \]

This is what we already derived in 1.5.0.
Consider a beam under pure bending conditions.

\[ u_i = \int \int \sigma \cdot \epsilon \, dA \, dl = \int \int \frac{M}{EI} \cdot \epsilon \, dA \, dl \]

Cross section

\[
\text{linear elastic materials} \quad = \frac{l}{2} \int \sigma_{xx} (y, z) \cdot \epsilon_{xx} (y, z) \, dA
\]

\[
= \frac{M_0^2 l}{2EI}
\]

\[ \Phi = \frac{1}{2} M_0 \theta. \]

\[ U_i = U_e \rightarrow \frac{M_0^2 l}{2EI} = \frac{1}{2} M_0 \theta. \]

\[ \theta_o = \frac{M_0 l}{EI} \]
(6) Consider a beam with complex axial and transverse bending:

\[ U_i = \int_a^b \left( \int x_i \, dx \right) + \int_a^b \int x_i \, dx \]

\[ = \int_a^b \frac{M(x)}{2EI} \, dx + \int_a^b \frac{N(x)}{2EA} \, dx \]

\[ \text{bending strain energy} \quad \text{axial strain energy} \]

\[ U_e = \frac{1}{2} \int (w(x) \cdot y 
 v(x)) \, dx + \frac{1}{2} \int (N(x) \cdot u(x)) \, dx + \frac{1}{2} F \cdot U(x) \]

\[ \text{external work of distributed transverse load} \quad \text{axial load} \quad \text{external work of concentrated loads} \]

\[ U_i = U_e \]

(7) Please note that we have neglected the strain energy due to shear at this stage...
Example: calculate the deflection in the beam using energy approach:

Conservation of

\[ U_e = \frac{1}{2} P \delta. \]

\[ U_i = \frac{1}{2EI} \int M^2 \, dx. \]

\[ = \frac{1}{2EI} \int_0^l P^2 x^2 \, dx. \]

\[ = \frac{P l^3}{6EI}. \]

\[ U_e = U_i \rightarrow \frac{P l^3}{6EI} = \frac{1}{2} P \delta \rightarrow s = \frac{Pl^3}{3EI}. \]

as we have shown through different methods.
Lecture 10: Principle of virtual work and unit load for trusses.

1. In this method, displacements are infinitesimal to keep the direction of the load constant.

2. Assume that a point mass is at equilibrium with some external forces. If a virtual displacement is applied to this point mass, the virtual external work is going to be zero.

\[ \sum_{i=1}^{n} P_i = 0 \rightarrow \sum_{i=1}^{n} P_i \cdot s_i = 0 \]

Equilibrium condition

virtual external work: \( W_{ext} \)

3. The same story is valid for a rigid body at equilibrium with external forces and moments.

4. For a deformable structure, after loading and deformation, the loads are at equilibrium.

Consider a \( dx \) element. This element may move, rotate, and deform after loading.

\[ dW_e = dW_r + dW_d = dW_d \]

the work due to rigid body motion

the work due to deformation

virtual work theorem for deformable structures.
for the deformable structure, the internal work can be stated as follows:

\[ \text{d} W_d = N \text{d}s + M \text{d}\theta + V \text{d}\lambda + T \text{d}\phi . \]

axial work  moment work  shearwork  torsion work

Therefore, the virtual work theorem for the entire structure can be stated as

\[
W_{\text{ext}} = \int_{\epsilon} \text{d}w_e = \int_{\epsilon} N \text{d}s + \int_{\epsilon} M \text{d}\theta + \int_{\epsilon} V \text{d}\lambda + \int_{\epsilon} T \text{d}\phi .
\]

The unit load method: in this method, we apply a unit load in the direction of interest in the virtual structure and consider a virtual displacement that is equal to the actual structure deformation and write the virtual work theorem for it.

The load and internal loads are real.

The load and internal load and displacements are virtual.
Virtual external work = virtual internal work

\[
1 \times \Delta = \int n \cdot ds + \int m \cdot d\theta + \int v \cdot d\alpha + \int f \cdot dp
\]

virtual load

actual displacement

virtual internal loads in structure 2.

\(\mathbf{8}\) If you are interested in displacement in a certain direction put a virtual force of 1 that gives rise to \(W_{\text{ext}} = 1 \times \Delta\). If you are interested in a rotation at a certain point, insert a virtual moment of 1 that gives rise to \(W_{\text{ext}} = J \times \theta\). If you are interested in how much 2 points get close to each other, insert 2 loads with different directions which gives rise to \(W_{\text{ext}} = 1 \times \Delta_{\text{app}}\).

\(\mathbf{9}\) Let's consider the case for the simple truss structure:

\[
\int n \cdot ds = \int \frac{Ndx}{EA} = \frac{nNl}{EA}
\]

\[
1 \times \Delta = \sum_{i=1}^{n_{\text{el}}} \frac{n_i \cdot N_i \cdot l_i}{E_i \cdot A_i}
\]

\(\{n_i\} : \text{force in } i^{\text{th}} \text{ element in virtual structure.}

\(\{N_i\} : \text{real } n_i\).
If there are actual settlement in supports, their contribution to the external work should be quantifiable.

\[
\begin{align*}
1 x \Delta + W_R &= \sum_i \frac{n_i N_i i_i}{E_i A_i} \\
W_R &= r_A \times \delta_A + r_B \times \delta_B + \ldots
\end{align*}
\]

\(r_A, r_B\) \rightarrow \text{virtual support forces in structure 2}.

\(\delta_A, \delta_B\) \rightarrow \text{actual support settlements in structure 1}.

If the temperature change happens in a structure, it affects the work done by the internal virtual loads.

\[
\int d\delta_i = x \cdot (AT)_i \cdot dx
\]

\[
\begin{align*}
\int n_i d\delta_i &= \int n_i x_i \cdot (AT)_i \cdot dx = n_i x_i (AT)_i \cdot l_i \\
\int n_i d\delta_i &= n_i \delta_i = \frac{n_i N_i l_i}{E_i A_i}
\end{align*}
\]

\(\Delta = \sum n_i x_i (AT)_i \cdot l_i + \sum \frac{n_i N_i l_i}{E_i A_i}\)

There might be an error during fabrication that might cause the free virtual internal work done by virtual works.

\[
1 x \Delta = \sum n_i (AL)_i
\]
The general unit load method for the entire truss structure can be stated as:

\[ 1 \times A + W_e = \sum_i \left\{ \frac{n_i N_i L_i}{E_i A_i} + n_i N_i (AT_i) L_i + n_i (AL_i) \right\} \]

Virtual internal work due to internal loads, temperature change, and fabrication error.

Example: find \( \delta^v_c \) and \( \delta^H_c \), \( EA = cte \).

\[ 1 \times \delta^H_c = \sum_i \frac{n_i N_i L_i}{EA} \]

\[ \delta^H_c = \frac{(P\sqrt{2})(\sqrt{2}) \times \sqrt{2}^2}{EA} + \frac{(-P)(-1) \ell}{EA} \]

\[ \delta^v_c = \sum_i \frac{n_i N_i L_i}{EA} \]

\[ \delta^v_c = \frac{P\ell}{EA} \left( 2\sqrt{2} + 1 \right) \]

\[ \delta^v_{el} = \frac{(-P)(-1) \ell}{EA} = \frac{P\ell}{EA} \]

\[ \delta_c = \sqrt{\delta^H_c^2 + \delta^v_{el}^2} \]
Example 15: Find $\delta_C^H$ if support at D settles $\frac{f}{100}$ downward.

\[ 1 \times \delta_C^H + W_R = \sum \frac{m_i N_i l_i}{EA} \]

\[ 1 + \delta_C^H = - \left( \frac{f}{100} \right) \times (1) = \frac{f}{100} \]

Example 16: Determine B and D get close to each other. (EA: Cle)

Superposition theorem:

\[ 1 \times \delta_{B/D} = \sum \frac{N_i m_i l_i}{EA} = \sum_i \left\{ \frac{n_i^1 N_i l_i}{EA} + \frac{n_i^2 N_i l_i}{EA} \right\} \]

(Practice the solution yourself)

Example 17: Determine the rotation of BC Element (EA: Cle)
\[ \theta_{BC} = \frac{\delta_B + \delta_C}{l} = \frac{1}{l_{BC}} \sum_i \left\{ \frac{n_i^1 N_i li}{EA} + \frac{n_i^2 N_i li}{EA} \right\} \]

\[ \geq \frac{1}{l_{BC} EA} \sum_i N_i (n_i^1 + n_i^2) li \]

Example: find \( \delta^H_C \) ducts temperature change.

\[ (AT)_i = \tau \]

\[ 1 \times \delta^H_C = \sum_i \alpha N_i (AT)_i li \]

\[ \delta^H_C = \alpha \tau \sum_i n_i li \]

\[ \delta^H_C = \alpha \tau \left( \sqrt{2} \times \sqrt{2} l + (-1) l \right) = \alpha \tau l \]
1. From previous sessions, we have the following relation for a general beam using virtual work theorem:

\[
W_e = W_{K} = \int_{0}^{l} n \, ds + \int_{0}^{l} m \, d\alpha + \int_{0}^{l} v \, d\xi + \int_{0}^{l} t \, d\eta
\]

\[
= \left( \int_{0}^{l} \frac{n N dx}{EA} \right) + \left( \int_{0}^{l} \frac{m M dx}{EI} \right) + \left( \int_{0}^{l} \frac{\alpha_s v V dx}{GA} \right) + \left( \int_{0}^{l} \frac{t T dx}{GJ} \right)
\]

\[
\frac{d\xi}{dx} = \frac{N dx}{EA}, \quad \frac{d\alpha}{dx} = \frac{M dx}{EI}, \quad \frac{d\eta}{dx} = \frac{\alpha_s v V dx}{GA}, \quad \frac{t T dx}{GJ}
\]

Actual beam structure 1:

Virtual beam structure 2:

\[\alpha_s\text{ is shear factor shape that can be found in...}\]

[Derivation of these relations can be found in more advanced classes and can be included if more time is allowed for entire set of lectures.]

2. Calculate \( \delta_{xx} \) in the structure below? (\( EA, GJ, EA, GA \text{ is known} \))
actual

\[ N = 0 \]
\[ V = -wL + wx \]
\[ M = wLx - wL^2 / 2 \]

virtual

\[ n = 1 \]
\[ v = -1 \]
\[ M = x \]

\[ n = +1 \]
\[ v = +1 \]
\[ M = x \]

plot the \( v, M \) diagrams:

\[ V : \]
\[ M : \]

\[ 1 \times \delta_c^H = 2 \int_0^L \frac{z (wLx - wx^2)}{EI} \, dx + 2 \int_0^L \frac{z (-1) (wx - wL^2)}{EI} \, dx \]

\[ \delta_c^H = \frac{5wL^4}{12EI} + \frac{\alpha_s wL^2}{GA} \]

- Bending
- Shear contribution
\[ \alpha = \frac{\text{Shear contr.}}{\text{bending contr.}} = \frac{\frac{6A}{\alpha s E^2}}{\frac{5}{12} \frac{w E^4}{EI}} = \frac{12 \alpha s E I}{5 G A \cdot I^2} \]

assuming a rectangular cross section: \((\alpha_s \approx 1.2)\), \((I = 0.3)\)

\[ \alpha = \frac{12 \times 1.2 \times E \times bh^3 \times 2(HV)}{12 \times 5 \times G \times bh \cdot I^2} \approx 0.6 \left(\frac{h}{I}\right)^2 \]

for beam \((\frac{h}{I}) < 0.1 \rightarrow [\alpha < 0.006]\)

This means that the effect of shear can be safely neglected in beams.

3. The effect of temperature in beams can be decomposed into mean temperature increase \((\frac{T_u + T_b}{2} - T_0)\) and the temperature difference between the top and the bottom, \(T_u\) and \(T_b\), respectively.

\[
\begin{align*}
T_0 & \rightarrow T_u \\
T_i & \rightarrow T_b
\end{align*}
\]

\[
\left\{ \begin{array}{l}
\delta s = \alpha \left( \frac{T_u + T_b}{2} - T_i \right) dx \\
\delta \theta = \alpha (T_b - T_u) \cdot dx
\end{array} \right. \rightarrow \begin{array}{l}
\int n \cdot \delta s = \int n \alpha \left( \frac{T_u + T_b}{2} - T_0 \right) dx \\
\int m \cdot \delta \theta = \int m \alpha (T_b - T_u) \cdot dx
\end{array}
\]
Therefore, we can extend the virtual work theorem for beams to include temperature effects:

\[ \Delta x \Delta + W_R = \int \frac{N M_{Ax}}{EA} + \int \frac{M_{Bx} M_{Bx}}{EI} + \int \frac{Q_{Ax} Q_{Ax}}{GA} + \int \frac{t T}{GJ} \, dx \]

\[ + \int n (\alpha T) \, dx + \int m \alpha (T_B - T_0) \, dx . \]

5. Determine the rotation at point C in the following structure. \( (T_0 = 0) \] [neglect the shear effects]

\[ \theta_c = 2 \int_0^l \frac{M_0 x}{E I} \, dx + \int_0^l \frac{Q}{E I} \, dx \]

\[ \theta_c = \frac{2 M_0}{E I l^2} \cdot \frac{x^3}{3} [l]_0^l + \left[ \frac{Q}{E I} \cdot \frac{x^2}{2} \right]_0^l \]

\[ \theta_c = \frac{2 M_0 l}{3 E I} + \frac{2 Q l}{E I} \]

**bending effect**  **temperature difference effect**
Determine the horizontal deflection at point B. \( \delta_B^h = ? \)

neglect the shear effects:

\[ M = -PR \sin \theta \]
\[ V = P \cos \theta \]
\[ N = -P \sin \theta \]

\[ m = -R(1 - \cos \theta) \]
\[ v = \sin \theta \]
\[ n = \cos \theta \]

\[ 1 \times \delta_B^h = \int_0^{\theta_1} \frac{nN \, ds}{EA} + \int_0^{\theta_1} \frac{mM \, ds}{EI} \]
\[ ds = r \, d\theta \]

\[ 1 \times \delta_B^h = \int_{\theta_0}^{\theta_1} \frac{P \cos \theta \cdot R \, d\theta}{EA} + \int_{\theta_0}^{\theta_1} \frac{PR^2 \sin \theta (1 - \cos \theta)}{EI} \, d\theta \]

\[ \delta_B^h = \frac{-PR}{2EA} \int_{\theta_0}^{\theta_1} \sin 2\theta \, d\theta + \frac{PR^2}{EI} \int_{\theta_0}^{\theta_1} \sin \theta (1 - \cos \theta) \, d\theta \]

\[ = -\frac{PR}{2EA} + \frac{PR^3}{2EI} \]

"axial contribution"  "bending moment contribution"
Determine the vertical deflection at point C for the below lattice structure. 

* neglect the shear terms and only consider the bending and torsional effects.

\[ \begin{align*}
BC: & \quad \begin{cases}
M = -Px \\
T = 0
\end{cases} \\
AB: & \quad \begin{cases}
M = -Pz \\
T = PL
\end{cases}
\end{align*} \]

\[ m = -x \quad \begin{cases}
+ = 0 \\
\frac{m}{l} = -z \\
t = P
\end{cases} \]

\[ 1 \times S_B^v = 2 \times \left( \int_0^l \frac{Pa^2}{EI} \, dx \right) + \int_0^l \frac{Ple}{GJ} \, dx \]

\[ S_B^v = 2 \frac{ple^3}{3EI} + \frac{ple^2}{GJ} \]

- bending effect
- torsional effect.
Lecture 12: Principle of minimum potential energy and Castigliano's theorem

1st Application in truss problems.

From the last lecture, we have the virtual work theorem for beams with arbitrary type of loading. Now, what if the loading in the virtual structure is set equal to the loading on the actual structure instead of one(1)? (note that entire loading from \( a \rightarrow p \) is considered)

\[
d\delta = \frac{Nd}{EA} \frac{x}{dx} \\
d\theta = \frac{M}{EI} \frac{dx}{dx} \\
d\lambda = \frac{asv}{GA} \frac{dx}{dx}
\]

The internal loads in virtual beam are equal to that of actual beam.

\[
W_{int} = U = \int N \delta + \int M \theta + \int V \lambda + \int T \phi
\]

\[
= \int \frac{N^2}{2EA} \frac{dx}{dx} + \int \frac{M^2}{2EI} \frac{dx}{dx} + \int \frac{asv^2}{GA} \frac{dx}{dx} + \int \frac{T^2}{6J} \frac{dx}{dx}
\]

Please note that the \( \frac{1}{2} \) factor comes from the assumption of linear elastic deformation in materials.

\[
W_{ext} = \sum \frac{1}{2} F_i \Delta_i + \sum \frac{1}{2} M_i \theta_i + \frac{1}{2} \int (\sigma \cdot \varepsilon) \, dx + \cdots \]

(distributed loads)
Please note the $\frac{1}{2}$ factor in external loading work comes from the condition of quasi-static loading.

2) Example: Calculate the deflection under the load via the energy method \((EI, GA = \text{cte})\)

\[
W_{\text{int}} = W_{\text{ext}}
\]

\[
2 \left( \int_0^L \frac{(P/2)^2}{2EI} \, dx + \int_0^L \frac{\alpha_5 (P/2)^2}{2 GA} \, dx \right) = \frac{1}{2} \, P \times S
\]

\[
\rightarrow \frac{1}{2} \, P \times S = \frac{P^2 \ell^3}{96EI} + \frac{\alpha_5 P^2 L}{8GA}
\]

\[
S = \frac{P^2 \ell^3}{48EI} + \frac{\alpha_5 P^2 L}{4GA}
\]

3) Consider the following body with the applied forces and constraints:

\[
W_{\text{int}} = W_{\text{int}} \left( S_1, S_2, \ldots, S_n \right)
\]

\[
dW_{\text{int}} = \frac{\partial W_{\text{int}}}{\partial S_1} \, dS_1 + \frac{\partial W_{\text{int}}}{\partial S_2} \, dS_2 + \ldots + \frac{\partial W_{\text{int}}}{\partial S_n} \, dS_n
\]

\[
dW_{\text{ext}} = P_1 \, dS_1 + P_2 \, dS_2 + \ldots + P_n \, dS_n
\]

According to conservation of energy: \(dW_{\text{int}} = dW_{\text{ext}}\)
Note that \( d\delta_1, d\delta_2, \ldots, d\delta_n \) are arbitrary small numbers. This means that their coefficients should be equal: Castigliano's theorem.

\[
\frac{\partial W_{\text{int}}}{\partial \delta_1} = P_1, \quad \frac{\partial W_{\text{int}}}{\partial \delta_2} = P_2, \quad \ldots, \quad \frac{\partial W_{\text{int}}}{\partial \delta_n} = P_n
\]

This means that the variation of internal strain energy with respect to deformation degrees of freedom is equal to the applied force in that point.

\((4)\)

We also can rearrange the term to derive what is known as the principle of minimum potential energy:

\[
\begin{align*}
\frac{dW_{\text{int}}}{\partial \Pi} & = \frac{dW_{\text{ext}}}{\partial \Pi} \\
\implies dW_{\text{int}} & = dW_{\text{ext}} \\
\implies d(W_{\text{int}} - W_{\text{ext}}) & = 0 \\
\implies d\Pi & = 0
\end{align*}
\]

\(\Pi = W_{\text{int}} - W_{\text{ext}}\)

Potential energy

\(\Pi\) should be minimum at equilibrium.

\[
\begin{align*}
\frac{d\Pi}{\partial \delta_1} & = 0, \quad \frac{d\Pi}{\partial \delta_2} = 0, \quad \ldots, \quad \frac{d\Pi}{\partial \delta_n} = 0
\end{align*}
\]

In practice, the above equation gives us a equation that we can use to derive \( n \) unknowns.
Example:

\[ \Delta = \frac{P \cdot l}{2 \cdot EA \cdot (2 \cos^3 \alpha + 1)} \]

- \( EA = \alpha \)
- due to symmetry D can only move in vertical direction.
- \( n = 3 + 3 \times 2 - 4 \times 2 = 1 \)
- the structure is indeterminate of \( \frac{1}{3} \) degree.

\[ \begin{aligned}
\Delta_1 &= \Delta_3 = \Delta \cos \alpha \\
\Delta_2 &= \Delta
\end{aligned} \]

\[ W_{\text{int}} = W_{\text{int}}^3 = \frac{1}{2} \cdot \frac{EA}{\ell_1} \cdot \delta_1 = \frac{1}{2} \left( \frac{EA \delta_1}{\ell_1} \right)^2 \]

\[ W_{\text{int}}^2 = \frac{1}{2} \cdot \frac{EA}{\ell} \cdot \delta^2 \]

\[ W_{\text{int}}^{\text{Tot}} = \sum_{i=1}^{3} W_{\text{int}}^i = \frac{EA \Delta^2}{2 \ell} (2 \cos^3 \alpha + 1) \]

- apply castigliano's theorem:

\[ \frac{\partial W_{\text{int}}^{\text{Tot}}}{\partial \Delta} = P \rightarrow \frac{EA \Delta}{\ell} (2 \cos^3 \alpha + 1) = P \]

\[ \Delta = \frac{PE}{EA (2 \cos^3 \alpha + 1)} \]

* Calculate forces in elements 1, 2 yourself at home..."
6. Find the displacement of joint A.

\[ n = 4 + 4 \times 2 - 5 \times 2 = 2 \]
the structure is 2 degrees indeterminate.

\[ \delta_1 = \Delta_1 \quad ; \quad \delta_2 = \frac{\sqrt{2}}{2} (\Delta_1 + \Delta_2) \quad ; \quad \delta_3 = \Delta_2 \]

\[ \delta_4 = \frac{\sqrt{2}}{2} (\Delta_2 - \Delta_1) \]

\[ W^1_{\text{int}} = \frac{EA}{2l} \Delta_1^2 \quad ; \quad W^2_{\text{int}} = \frac{EA}{2l} \frac{(\Delta_1 + \Delta_2)^2}{2} \]

\[ W^3_{\text{int}} = \frac{EA}{2l} \Delta_2^2 \quad ; \quad W^4_{\text{int}} = \frac{EA}{2l} \frac{(\Delta_2 - \Delta_1)^2}{2} \]

\[ W^{\text{tot}}_{\text{int}} = \sum_{i=1}^{4} W^i_{\text{int}} = \frac{EA}{4l} \left( 2\Delta_1^2 + 2\Delta_2^2 + (\Delta_1 + \Delta_2)^2 + (\Delta_2 - \Delta_1)^2 \right) \]

\[ = \frac{EA}{4l} \left( 4\Delta_1^2 + 4\Delta_2^2 + 2\Delta_1 \Delta_2 - 2\Delta_1 \Delta_2 \right) = \frac{EA}{l} \left( \Delta_1^2 + \Delta_2^2 \right) \]

Castigliano's theorem:

\[ \frac{\partial W^{\text{tot}}_{\text{int}}}{\partial \Delta_1} = \frac{EA}{l} (2\Delta_1) = 0 \quad \rightarrow \quad \Delta_1 = 0 \]

\[ \frac{\partial W^{\text{tot}}_{\text{int}}}{\partial \Delta_2} = \frac{EA}{l} (2\Delta_2) = P \quad \rightarrow \quad \Delta_2 = \frac{Pe}{2EA} \]

Calculate the internal forces yourself at P.
Example: calculate the nodal displacements \( \left( \frac{EA}{L} = cL \right) \).

\[ n = 6 + 4 - 4 \times 2 = 2 \]

"2 degrees indeterminate"

\[
\delta_1 = \Delta_1 \\
\delta_2 = \Delta_2 \\
\delta_3 = \frac{\sqrt{2}}{2} (\Delta_1 + \Delta_4) \\
\delta_4 = \frac{\sqrt{5}}{2} (\Delta_3 + \Delta_2) \\
\delta_5 = \Delta_4 \\
\delta_6 = \Delta_3
\]

\[
W_{\text{int}}^1 = \frac{EA}{2L} \Delta_1^2 \\
W_{\text{int}}^2 = \frac{EA}{2L} \Delta_2^2 \\
W_{\text{int}}^3 = \frac{EA}{4L} (\Delta_1 + \Delta_4)^2 \\
W_{\text{int}}^4 = \frac{EA}{4L} (\Delta_3 + \Delta_2)^2 \\
W_{\text{int}}^5 = \frac{EA}{2L} (\Delta_4)^2 \\
W_{\text{int}}^6 = \frac{EA}{2L} (\Delta_3)^2
\]

\[
W_{\text{tot}} = \sum_{i=1}^{6} W_{\text{int}}^i = \frac{EA}{4L} \left( 2\Delta_1^2 + 2\Delta_2^2 + (\Delta_1 + \Delta_4)^2 + (\Delta_3 + 2\Delta_2)^2 + 2\Delta_4^2 + 2\Delta_3^2 \right)
\]

Castigliano's theorem:

\[
\frac{\partial W_{\text{tot}}}{\partial \Delta_1} = \frac{EA}{4L} (6\Delta_1 + 2\Delta_4) = 0 \\
\frac{\partial W_{\text{tot}}}{\partial \Delta_2} = \frac{EA}{4L} (6\Delta_2 + 2\Delta_3) = P \\
\frac{\partial W_{\text{tot}}}{\partial \Delta_3} = \frac{EA}{4L} (4\Delta_3 + 2(\Delta_2 + \Delta_3)) = 0 \\
\frac{\partial W_{\text{tot}}}{\partial \Delta_4} = \frac{EA}{4L} (6\Delta_4 + 2\Delta_1) = 0
\]
\[
\frac{EA}{4l} \begin{bmatrix}
46 & 0 & 0 & 2 \\
0 & 62 & 0 & 0 \\
0 & 0 & 26 & 0 \\
6 & 0 & 0 & 6 \\
\end{bmatrix} \begin{bmatrix}
\Delta_1 \\
\Delta_2 \\
\Delta_3 \\
\Delta_4 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
P \\
0 \\
0 \\
\end{bmatrix}
\]

Stiffness matrix \quad \text{displacement vector} \quad \text{force vector}

You can solve the above system of linear equations, by simply inverting it:

\[
\Delta_1 = \Delta_4 = 0, \quad \Delta_2 = \frac{3Pl}{4EA}, \quad \Delta_3 = \frac{-pl}{4EA}
\]
Lecture 13: Castigliano’s theorem and its application in beams and frames:

1. From the previous lecture, we know that:

\[
\frac{\partial W_{\text{int}}}{\partial \varepsilon_1} = P_1, \quad \frac{\partial W_{\text{int}}}{\partial \varepsilon_n} = P_n
\]

where the \( W_{\text{int}} \) is the internal strain energy stored in the deformable body.

\[
W_{\text{int}} = \int \frac{N^2}{2EA} \, dx + \int \frac{M^2}{2EI} \, dx + \int \frac{\alpha_3 V^2}{2GA} \, dx + \int \frac{T^2}{\phi J} \, dx
\]

2. In practice, we always try to express \( W_{\text{int}} \) in terms of the deformation degrees of freedom in the structure, \( W_{\text{int}} = W_{\text{int}}(\varepsilon_1, \ldots, \varepsilon_n) \).

3. How does the internal energy look like for a beam in terms of the displacement and rotation of its end points?

where I have used the superposition theorem to decompose the structure with three displacements to three structures with one displacement.

4. Let's use the conjugate beam method to calculate internal forces in structures 1 and 3 in terms of \( \theta_1 \) and \( \theta_2 \). Please note that the internal forces in structure 2 are similar to that of 1.
5. Conjugate beam

\[ \sum M_A' = 0 \rightarrow \frac{M_1 l}{2EI} \frac{E}{3} = \frac{M_2 l}{2EI} \frac{E}{3} \]

\[ \rightarrow M_1 = 2M_2 \]

\[ \sum F_y' = 0 \rightarrow \theta_1 + \frac{M_2 l}{2EI} - \frac{M_1 l}{2EI} = 0 \]

\[ \rightarrow M_2 = \frac{2\theta_1 EI}{l} \quad \& \quad M_1 = \frac{4\theta_1 EI}{l} \]

Similarly:

\[ M_1 = \frac{2\theta_2 EI}{l} \quad \& \quad M_2 = \frac{4\theta_2 EI}{l} \]

\[ M_1 = M_2 = -\frac{6EI}{l^2} \]

6. Now, by superposing all these solutions we have:

\[ M_1 = \frac{2EI}{l} (2\theta_1 + \theta_2 - 3\delta/l) \]

\[ M_2 = \frac{2EI}{l} (\theta_1 + 2\theta_2 - 3\delta/l) \]
\[
\begin{align*}
R_1 &= \frac{6EI}{\ell^2} \left( \theta_1 + \theta_2 - 2\theta / \ell \right) \\
R_2 &= \frac{-6EI}{\ell^2} \left( \theta_1 + \theta_2 - 2\theta / \ell \right)
\end{align*}
\]

(7) Knowing that external and internal energies are equal, we can write:

\[
W_{\text{ext}} = W_{\text{int}} = \frac{1}{2} M_1 \theta_1 + \frac{1}{2} M_2 \theta_2 + \frac{1}{2} R_2 S + \frac{1}{2} N \Delta + \frac{1}{2} T \phi
\]

\[
= \frac{2EI}{\ell} \left( \theta_1^2 + \theta_1 \theta_2 + \theta_2^2 \right) - \frac{6EIS}{\ell^2} \left( \theta_1 + \theta_2 \right) + \frac{6EI}{\ell^3} S^2
\]

\[+ \frac{EA}{2\ell} \left( A \right)^2 + \frac{GJ}{2\ell} \phi_2^2 \]

Please note that:

\[s = \Delta_3 - \Delta_1\]

\[\Delta = \Delta_4 - \Delta_2\]

\[\phi = \phi_1 - \phi_2\]

This means that displacements are relative.

(8) Example: Derive the internal force in support \( B \) in terms of \( M_B \).

There is only \( \theta_B \) in this structure and \( \theta_A \) and \( S \) are equal to zero.

\[
W_{\text{int}} = \frac{2EI}{\ell} \theta_2^2 \rightarrow \frac{4EI}{\ell} \theta_B = M_B
\]

\[\theta_B = \frac{M_B \ell}{4EI}\]

\[R_B = -\frac{6EI}{\ell^2} \left( \frac{M_B \ell}{4EI} \right) = -\frac{3M_B}{a\ell}\]

(69)
Example: Determine $\theta_B$ in the following structure (neglect axial deformation).

\[ \begin{align*}
\psi_{int}^1 &= \frac{2EI}{\ell} \theta_B^2 \quad \Rightarrow \quad \psi_{int}^2 = \frac{8EI}{\ell} \theta_B^2 \\
\frac{\partial \psi_{int}}{\partial \theta_B} &= \frac{16EI}{\ell} \theta_B = -M_0 \\
\theta_B &= -\frac{M_0}{16EI} 
\end{align*} \]

You can use $\theta_B$ to calculate internal forces and plot moment & shear diagrams.

Example: Calculate the nodal displacement:

\[ \begin{align*}
\psi_{int}^1 &= \frac{2EI}{\ell} \theta_B^2 - \frac{6EI}{\ell^2} A_2 \theta_B + \frac{6EI A_2^2}{\ell^3} + \frac{E A A_2^2}{2\ell} \\
\psi_{int}^2 &= \frac{2EI}{\ell} \theta_B^2 - \frac{6EI}{\ell^2} (A_1) \theta_B + \frac{6EI (-A_1)^2}{\ell^3} + \frac{E A A_2^2}{2\ell} \\
\psi_{int}^{tot} &= \frac{4EI}{\ell} \theta_B^2 - \frac{6EI}{\ell^2} \theta_B (A_2-A_1) + \left( \frac{6EI}{\ell^2} + \frac{EA}{2\ell} \right) (A_1^2 + A_2^2) \\
\frac{\partial \psi_{int}^{tot}}{\partial A_1} &= \frac{6EI}{\ell^2} \theta_B + 2A_1 \left( \frac{6EI}{\ell^2} + \frac{EA}{2\ell} \right) = 0 \\
\frac{\partial \psi_{int}^{tot}}{\partial A_2} &= -\frac{6EI}{\ell^2} \theta_B + 2A_2 \left( \frac{6EI}{\ell^2} + \frac{EA}{2\ell} \right) = 0 \\
\frac{\partial \psi_{int}^{tot}}{\partial \theta_B} &= \frac{8EI}{\ell} \theta_B - \frac{6EI}{\ell^2} (A_2-A_1) = -M_0 
\end{align*} \]
Example: Calculate the nodal displacements and rotations. Neglect the axial deformations for simplicity.

\[
W_{AB} = \frac{2EI}{E} \theta_B^2 + \frac{6EIS}{E} \theta_B \theta_C + \frac{6EIS}{E^2} \theta_C^2
\]
\[
W_{BC} = \frac{2EI}{E} \left( \theta_B^2 + \theta_B \theta_C + \theta_C^2 \right)
\]
\[
W_{CD} = \frac{2EI}{E} \theta_C^2 + \frac{6EIS}{E^2} \theta_B + \frac{6EIS}{E^2} \theta_C + \frac{12EIS}{E^2} \theta_B \theta_C
\]

\[
\omega \frac{\text{rot}}{\text{tot}} = \frac{2EI}{E} \left( 2 \theta_B^2 + \theta_B \theta_C + 2 \theta_C^2 \right) + \frac{6EIS}{E^2} \left( \theta_B + \theta_C \right) + \frac{12EIS}{E^2} \theta_B \theta_C
\]

\[
\delta = \frac{SEI}{EI} \frac{pE^3}{84}
\]
\[
\theta_B = \theta_C = -\frac{pE^2}{28EI}
\]
you can calculate the support forces knowing the kinematic degrees of freedom. The results are as follows:

\[ \begin{align*}
\text{P} & \\
\frac{P}{2} & \leftarrow \sqrt{P^{2} - \frac{2PL}{7}} \\
\frac{3P}{7} & \\
\frac{P}{2} & \leftarrow \frac{2PL}{7} \\
\frac{3P}{7} &
\end{align*} \]
Lecture 14: 2nd Castigliano's theorem and its applications

1. Consider a linear elastic structure:

\[ dP_1 = dP_2 = dP_3 = \ldots = dP_{n-1} = 0 \]

\[ W_{int} = W_{int}(P_1, P_2, \ldots, P_n) \]

\[ dW_{int} = \frac{\partial W_{int}}{\partial P_1} dP_1 + \frac{\partial W_{int}}{\partial P_2} dP_2 + \ldots + \frac{\partial W_{int}}{\partial P_n} dP_n \]

\[ = \frac{\partial W_{int}}{\partial P_n} dP_n : \text{the change in the internal energy by changing } P_n \text{ by } dP_n \]

2. 

\[ dW_{ext} = P_n \cdot d\delta_n = dP_n (d\delta_n + S_n) = dP_n \frac{d\delta_n}{\text{small value}} + S_n dP_n \]

The above relation comes from the fact that the structure is linear elastic and therefore the order of applying \( P_n \) and \( dP_n \) can be changed.

\[ P_n \cdot d\delta_n = S_n \cdot dP_n \]

Considering 1) and 2), we have:

\[ dW_{int} = dW_{ext} \quad \Rightarrow \quad \frac{\partial W_{int}}{\partial P_n} \cdot dP_n = S_n \cdot dP_n \]

\[ \frac{\partial W_{int}}{\partial P_n} = S_n \]
2. Let's apply Castigliano's theorem to trusses:

\[ W_{int} = \sum_{i} \frac{N_i \cdot E_i}{2 E_i A_i} \rightarrow \delta_n = \sum_{i} \frac{\partial N_i}{\partial N_i} \frac{\partial^2 N_i}{\partial P_i} \frac{E_i A_i}{E_i A_i} \]

3. Calculate the horizontal deflection at point Band C. Also determine how much B gets close to D. (EA=const)

\[ \delta_B^H = \sum_{i} \frac{\partial N_i}{\partial P_i} \frac{\partial^2 N_i}{\partial P_i} \frac{E_i A_i}{E_i A_i} \]

\[ = \frac{(P - P)(1)}{EA} + \frac{(P - P)(-1)}{EA} + \frac{(P)(\sqrt{2})(\sqrt{2})P^2}{EA} = \frac{PE}{2EA} (4 + 4\sqrt{2}) \]

\[ = \frac{PPE}{EA} (1 + 2\sqrt{2}) \]

\[ \delta_C^H = \sum_{i} \frac{N_i \cdot \partial^2 N_i}{\partial P_i \partial Q} \frac{E_i}{E_i A_i} \]

\[ = \frac{\sqrt{2}(P+Q) \cdot \sqrt{2} + \sqrt{2}}{EA} + \frac{-(P+Q)(-1)}{EA} \]

\[ = \frac{(P + Q)l}{EA} (2\sqrt{2} + 1) \]

\( \delta_C^H \) is zero in the structure:

\[ \rightarrow \delta_C^H = \frac{PL}{EA} (2\sqrt{2} + 1) \]
\[ S_{B/P} = \sum_{i} \frac{N_i \left( \frac{\partial N_i}{\partial Q} \right) \frac{Q}{E_i A_i}}{E+1} \]

\[ = \frac{F \cdot Q \cdot (1 - \frac{F}{2})}{EA} \cdot \frac{1}{E+1} \]

\[ + \frac{-(P \cdot Q \cdot \frac{F}{2}) \cdot (1 - \frac{F}{2})}{EA} \cdot \frac{1}{E+1} \]

\[ + \frac{-(P \cdot Q \cdot \frac{F}{2}) \cdot (1 - \frac{F}{2})}{EA} \cdot \frac{1}{E+1} \]

Noting that \( Q \) is zero in the actual structure:

\[ S_{B/P} = \frac{P \cdot E}{E+1} \]

4) Let's apply and Castigliano's theorem to beams and frames:

\[ W_{int} = \int \frac{N^2}{2EA} + \int \frac{M^2}{2EI} + \int \frac{\alpha_s V^2}{2G A} + \int \frac{T^2}{2G J} \, dx \]

\[ S_n = \int \frac{N \frac{\partial N}{\partial P_n}}{EA} \, dx + \int \frac{M \frac{\partial M}{\partial P_n}}{EI} \, dx + \int \frac{\alpha_s V \frac{\partial V}{\partial P_n}}{G A} \, dx + \int \frac{T \frac{\partial T}{\partial P_n}}{G J} \, dx \]

\[ \Theta_n = \int \frac{N \frac{\partial N}{\partial M_n}}{EA} \, dx + \int \frac{M \frac{\partial M}{\partial M_n}}{EI} \, dx + \int \frac{\alpha_s V \frac{\partial M}{\partial M_n}}{G A} \, dx + \int \frac{T \frac{\partial T}{\partial M_n}}{G J} \, dx \]

Note that if there are no point load and moment applied at the location of interest, we can put an arbitrary force and at the end, put its value to zero.
Example: Calculate the horizontal, vertical and the rotation of point C.

Please note that the beams and columns are short and therefore shear contributions cannot be neglected.

BC: \[
\begin{aligned}
M &= -M_0 - Po x - \frac{wx^2}{2} \\
V &= Po + wx \\
N &= Q_0
\end{aligned}
\]

AB: \[
\begin{aligned}
M &= -M_0 - Pe - Q_0 x - \frac{wle^2}{2} - \frac{wx^2}{2} \\
V &= Q_0 + wx \\
N &= -we -Po
\end{aligned}
\]

\[
\delta V_c = \frac{\partial W_{int}}{\partial P_o} = \int M \cdot \frac{\partial M_o}{\partial P_o} \frac{dx}{EI} + \int N \cdot \frac{\partial N}{\partial P_o} \frac{dx}{EA} + \int N \cdot \frac{\partial N}{\partial P_o} \frac{dx}{EA}
\]

\[
= \int_0^\ell (-M_o - Po x - \frac{wx^2}{2}) (-x) \frac{dx}{EI} + \int_0^\ell \gamma (P_o + wx)(1) \frac{dx}{EA} + \int_0^\ell Q_0 (0) \frac{dx}{EA}
\]

\[
+ \int_0^\ell (-M_o - Pe - Q_0 x - \frac{we^2}{2} - \frac{wx^2}{2}) (-x) \frac{dx}{EI} + \int_0^\ell \gamma (P_o + wx)(0) \frac{dx}{EA} + \int_0^\ell \gamma (-weP_o) \frac{dx}{EA}
\]

\[
(-1) \cdot \frac{dx}{EA}
\]
Assuming $Q_0$, $P$, and $M_0$ to be zero:

\[ S^V_C = \int_0^L \frac{\omega x^3}{2EI} \, dx + \int_0^L \frac{\omega x^2}{6A} \, dx + \int_0^L \frac{\omega + \omega x^2}{2EI} \, dx + \int_0^L \frac{\omega x^4}{EA} \, dx \]

\[ \rightarrow S^V_C = \frac{19\omega x^4}{24EI} + \frac{\omega x^2}{2GA} + \frac{\omega x^2}{EA} \]

\[ S^H_C = \frac{\partial W_{ut}}{\partial Q_0} = \int_0^L \frac{\omega x^2}{EI} \, dx + \int_0^L \frac{\omega x^2}{GA} \, dx + \int_0^L \frac{\omega x}{EA} \, dx \]

\[ + \int_0^L \frac{\omega x^2}{EI} \, dx + \int_0^L \frac{\omega x^2}{GA} \, dx + \int_0^L \frac{\omega x}{EA} \, dx \]

\[ \rightarrow S^H_C = \frac{\omega x^4}{8EI} + \frac{\omega x^2}{2GA} \]

\[ \theta_C = \frac{\partial W_{ut}}{\partial M_0} = \int_0^L \frac{\omega x^2}{2EI} \, dx + \int_0^L \frac{\omega x^2}{2EI} \, dx \]

\[ \theta_C = \frac{5}{6} \frac{\omega x^3}{EI} \]
Lecture 15: Influence line in beams for determinate cases

support forces and moments, internal shear and moment.

1. The internal and support forces in structural systems are functions of the magnitude and location of external forces inserted to the system. The purpose of influence line is to characterize the effect of loading location on the forces generated in the structure and its supports.

2. The simplest case for a moving load on a structure is cantilever. The question is: "How does the moment at the fixed end change as a function of the location of load?"

\[ M = -1x(l-x) = x-l \]

Very similar to shear and moment diagram, we can plot the influence line of the unit load generated moment at A:

\[ -e \]
3. Now, let's imagine instead of unit force, we apply load $F$ at $C$:

$$M_A = F \times \left(-\frac{2l}{3}\right) = -\frac{2Fl}{3}$$

4. The next step would be considering a load system which is a combination of three loads:

$$M_A = -\frac{4Fl}{5} - \frac{3P_2l}{5} - \frac{2P_1l}{5}$$

5. The procedure is the same for the distributed load system.

$$dM_A = (w \, dx) \left( x - e \right)$$

$$= (w \, dx) \left( x - e \right)$$

$$M_A = \int_{e/4}^{e/2} w(x - e) \, dx = \frac{w l^2}{2} - \frac{w l^2}{4} \left[ \frac{e}{2} - e \right]$$

$$\frac{wl^2}{8} - \frac{wl^2}{2} - \frac{wl^2}{32} + \frac{wl^2}{4} = -\frac{5wl^2}{32}$$
6) Example: calculate the influence line of the internal force in the cable CD:

\[ \Sigma M_A = 0 \rightarrow F \cdot \sin \alpha \left( \frac{e}{2} \right) - 1 \cdot x \left( \frac{e-x}{2} \right) = 0 \]

\[ \rightarrow F = \frac{e-x}{\frac{e}{2} \cdot \sin \alpha} = \frac{2(1 - \frac{x}{e})}{\sin \alpha} \]

7) The influence line of support forces can be calculated the same way:

\[ R_B \cdot e - 1 \cdot x \cdot \frac{e}{2} = 0 \]

\[ \rightarrow R_B = \frac{x}{e} \]

\[ R_C = 1 - \frac{x}{e} = \frac{e-x}{e} \]

8) From the above example, we find that we can calculate the influence line of internal support forces, by removing that support and moving it in the positive direction for "1" displacement, and plot the rest of deformations consistently.
(a) Example: plot the influence line for $R_A$, $R_B$, $R_C$ and $R_D$.

(b) The influence line of shear forces can be calculated as follows:

$$
\begin{cases}
    v = R_A & \frac{-a}{2} < x < \frac{e}{2} \\
    v = -R_B & \frac{e}{2} < x < -\frac{a}{2}
\end{cases}
$$
To plot the shear influence line at a point, add shear roller to that particular point and move the sides in accordance with positive conventions. Make sure that the two sides of shear influence line should be parallel and the sum should go to one of their difference.

Example: plot the shear influence lines at 1-5 sections.

The influence line of bending moment can be calculated as follows:
\[
\begin{align*}
&M = R_A (l-a) \quad -a < x < \frac{l}{2} \\
&M = R_B (a) \quad -\frac{3}{2} l < x < -a
\end{align*}
\]

complementary angles:
\[
\hat{\theta} = \hat{A} + B
\]

\[
\tan (\hat{\theta}) = \tan (\hat{A} + \hat{B}) = \frac{\tan (\hat{A}) + \tan (\hat{B})}{\frac{a}{\ell} + \frac{e-\ell}{\ell}}
\]

\[
\tan (\hat{\theta}) = 1 \quad \tan (\hat{\theta}) = \hat{\theta} = 1
\]

(15) Therefore, to plot the influence line of the moment at an arbitrary point, we can add a hinge to that particular point and move it upward and plot the deformation of the beam. Please note that the angle change should be \( \frac{\pi}{2} \) at the location that we have inserted the hinge.

(16) Example: plot the influence line of bending moment at the designated locations on the determined multispant beam below:
1. As discussed in the previous lecture, to plot the influence line for internal and support forces, we have to remove the resistance against that force or moment and move it in the positive direction or rotation with a unit magnitude.

2. The procedure explained above holds true for both determined and indeterminate beams.

3. Example: Plot the influence line of $RB$, $RC$, $V_m$ and $M_m$.

\[ n = 7 \]

\( \Rightarrow 7 \text{ degrees indeterminate.} \)
4) Example: plot the influence line of the support moment for the double fixed-end beam:

5) Therefore, we can see that the approximate influence line in indeterminate beams is nothing but approximate deformation of beams. For that we only have to consider boundary conditions.

6) Occasionally, we design floor systems that concrete slab lay on top of floor beams and the floor beams are connected to the girders, as shown in the figure below:

7) In fact, the function of floor beams is to translate the moving load to the girder.
In determinate beams, the influence line of support forces are not affected by the floor beam and stays the same as the regular beam. However, the influence line of the shear and moment changes depending on the distance between floor beams.

Example: Plot the shear and moment influence line at section m-n in the following simply supported beam.

![Diagram of a simply supported beam with sections labeled a, b, c, and d, and support points labeled A and B.]

\[
\begin{align*}
V_{mn} &= -R_B \quad 0 < x < l \\
V_{mn} &= +R_A \quad 2l < x < 4l \\
V_{mn} &= \frac{y}{E I} f_B + \frac{l - y}{E I} f_c \quad l < x < 2l.
\end{align*}
\]
To plot the shear influence line for floor beams and girders, we plot the influence line at the two sides of the floor beams and connect the ends via a line.

The moment at different regions are:

\[
\begin{align*}
M_{mn} &= R_B (2l - a) \quad 0 < x < l \\
M_{mn} &= R_A (2l - a) \quad l < x < 4l \\
M_{mn} &= f_b \cdot \frac{y}{l} + f_c \cdot \frac{l - y}{l} \quad l < x < 2l.
\end{align*}
\]

To plot the influence line for moment, we plot the influence lines of the two regions and connect them via a straight line.
The influence line of support forces and internal load can be readily calculated by applying the unit load to the node points and calculating the relevant forces, and adding to the plot one by one. Subsequently, we can connect the dots via lines as if the lines are connected via a member then the forces are going to be distributed linearly in between them.

Plot the influence line for $R_A$, $R_B$ and $F_{DB}$:
Lecture 17: Computer analysis of trusses using Matlab.

01. This is the process for the computer analysis of structures:

01. Read the geometry file:
   - nodes, elements, load and boundary conditions.

02. Assemble the stiffness
    matrix of the structure in global coordinate system.

03. Perform Static Condensation

04. Calculate Nodal displacements

05. Calculate Reaction Forces.

06. Calculate internal forces
function Analyze Truss

% Main code for solving 2D Truss problems using stiffness method. The code
% reads the geometry of the structure (nodal positions and connectivity),
% elastic modulus, member cross-section area, boundary conditions and
% assembles the stifness matrix, nodal force and displacement vectors. It
% subsequently performs the static condensation and finds the nodal
% displacements, reaction forces and internal forces. It also plots the
% undeformed and deformed trusses.
% cleaning the screen and closing all the figures (new start)
clear; close all;

% Read the truss input file using t
[nodes, elems, E, A, bcs, loads]=Input_Truss;
Nel = size(elems,1);
% Nel : number of truss elements
Nnodes = size(nodes,1);
% Nnode : number of nodal points

% decide degrees of freedom + Initiate Matrices
% Note: Degrees of freedom corresponding to node "i" are:
% [2*(i-1)+1 2*(i-1)+2]

% alldofs: total number of degrees of freedom = 2*Nnodes
alldofs = 1:2*Nnodes;
% K : global stiffness matrix
K = zeros(2*Nnodes,2*Nnodes);
% u : global displacement vector
u = zeros(2*Nnodes,1);
% f : global force vector
f = zeros(2*Nnodes,1);

% Create a list of all boundary conditions and relevant displacements
% dofspec : a vector containing specified dofs (Boundary Conditions:BC)
dofspec = [];
for ii = 1:size(bcs,1)
    % thisdof: the degree of freedom that pertains to ii BC
    thisdof = 2*(bcs(ii,1)-1)+bcs(ii,2);
    % adding thisdof to the list of all boundary conditions (dofspec)
    dofspec = [dofspec thisdof];
    % adding the value of the displacement in the BCs to the displacemet vector
    u(thisdof)=bcs(ii,3);
end
% doffree : the list of all degree of freedom that are free to move
doffree = alldofs;
% Delete specified dofs from all dofs
doffree(dofspec) = [];

% Create the global load vector
for ii = 1: size(loads,1)
    % forcedof: the degree of freedom at which the force is implemented
    forcedof = 2*(loads(ii,1)-1)+loads(ii,2);
    % adding each force to the force vector
    f(forcedof)=loads(ii,3);
end

% Initialize the global stiffness matrix
for iel = 1:Nel
    % elnodes : the nodes at the either side of element iel
    elnodes = elems(iel,1:2);
    % nodexy : coordinates of the nodes of element iel
    nodexy = nodes(elnodes,:);
    % Get the element global stiffness matix for the current element
    [Kel] = TrussElement2D(nodexy, E(iel), A(iel));
function [Kel] = TrussElement2D(nodexy, E, A)
% This function must return a 4x4 element stiffness matrix.
% This matrix must be in global coordinate system. The inputs are:
% nodexy : [ x1 y1; x2 y2]; E : Young's Modulus; A : Cross Section Area.

% DeltaXY: DeltaX and DeltaY of the two joint of the element
DeltaXY = [ (nodexy(2,1)-nodexy(1,1)) (nodexy(2,2)-nodexy(1,2)) ];
% L : Length of the element
L = norm(DeltaXY);
% CosSin(1) = cos(theta); and CosSin(2) = sin(theta);
CosSin = DeltaXY / L;

% Kel_axial : local stiffness matrix
Kel_axial = E*A/L*[1 -1; -1 1];

% Tmatrix: Rotation matrix that transforms from local to global coordinates
Tmatrix = [CosSin(1) CosSin(2) 0 0; 0 0 CosSin(1) CosSin(2)];
% Kel : Global stiffness matrix of the element
Kel = Tmatrix'*Kel_axial*Tmatrix;

end
function [nodes, elems, E, A, bcs, loads]=Input_Truss
% Example of a determinate truss with five elements and 4 nodes.

% nodes matrix contains x and y coordinate of the truss nodes
% First Column : x coordinate; Second Column : Y coordinate
nodes=[0.0 0.0; ... 
    2.0 0.0; ... 
    2.0 2.0; ... 
    0.0 2.0; ];

% eleme matrix contains the connectivity of the nodes
% First column : node i; Second coulm node j; i and j are connected via an
% element
elems=[1 2; ... 
    2 3; ... 
    3 4; ... 
    4 1; ... 
    1 3];

% E matrix contains elastic modulus of each element
E = [2e11; 
    2e11; 
    2e11; 
    2e11; 
    2e11];

% A matrix contains cross-section area of each element
A = [1e-2; 
    1e-2; 
    1e-2; 
    1e-2; 
    1e-2];

% bcs matrix contains the specified boundary conditions
% First column : node, Second Column : DOF (1 for X and 2 for Y)
% Third column : the displacement in that DOF
bcs = [1 1 0; ... 
    1 2 0; ... 
    2 2 0];

% loads matrix contains nodal forces
% First Column : Nodes; Second Column DOS (1 for X and 2 for Y)
% Third Column : Forces
loads = [3 1 10^4];
end
Lecture 18: Computer analysis of trusses using stiffness method:

1. The idea behind stiffness method is finite element analysis in which a structure is decomposed into members and nodes. The nodes, joints in the truss structure, are connected by members. Each member adds stiffness to the overall stiffness of the structure, known as global stiffness matrix.

   \[ \text{global stiffness matrix} \rightarrow K_g \rightarrow \text{global displacement vector} \rightarrow \mathbf{u} = F \rightarrow \text{global force vector} \]

2. \( \mathbf{F} \) contains all the external nodal forces exerted on the structure. \( \mathbf{u} \) contains all the nodal displacements in the truss, and \( K_g \) is global stiffness of truss structure assembled by adding individual stiffness of each element \( K_e \).

3. The unknowns of our problem, the global displacement vector \( \mathbf{u} \), can be found simply by inverting global stiffness matrix:

   \[ \mathbf{u} = K_g^{-1} \mathbf{F} \]

4. The nodal displacements for each element can be used to calculate its internal forces.
(5) We begin with constructing the local stiffness matrix of a truss member in its own local coordinate system (X-Y):

\[ F_A = \left( \frac{EA}{l} \right)_i \delta_A - \left( \frac{EA}{l} \right)_i \delta_B \]

\[ F_B = \left( \frac{EA}{l} \right)_i \delta_B - \left( \frac{EA}{l} \right)_i \delta_A \]

\[ \rightarrow \left( \frac{EA}{l} \right)_i \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \delta_A \\ \delta_B \end{bmatrix} = \begin{bmatrix} F_A \\ F_B \end{bmatrix} \]

Local stiffness matrix

(6) Now, let's try to express the local stiffness matrix in terms of the global coordinate system (X-Y): we start with displacements.

\[ \delta_A = \delta_1 \cos \theta + \delta_2 \sin \theta \]

\[ \delta_B = \delta_3 \cos \theta + \delta_4 \sin \theta \]

\[ \begin{bmatrix} \delta_A \\ \delta_B \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ 0 & 0 \cos \theta \sin \theta \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \]

\[ \text{Local: } \delta_A \text{ and } \delta_B \text{ are displacements in local coordinates, while } \delta_1, \delta_2, \delta_3 \text{ and } \delta_4 \text{ are nodal displacements in global coordinates.} \]
Now, let's deal with the forces between local and global coordinates:

\[ F_A^x = F_A \cos \theta \]
\[ F_A^y = F_A \sin \theta \]
\[ F_B^x = F_B \cos \theta \]
\[ F_B^y = F_B \sin \theta \]

\[
\begin{bmatrix}
F_A^x \\
F_A^y \\
F_B^x \\
F_B^y
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & 0 & 0 & 0 \\
0 & \sin \theta & 0 & 0 \\
0 & 0 & \cos \theta & 0 \\
0 & 0 & 0 & \sin \theta
\end{bmatrix}
\begin{bmatrix}
F_A \\
F_B
\end{bmatrix}
\]

We can plug the global terms into the local stiffness relation:

\[
\left( \frac{EA}{L} \right)_i \begin{bmatrix}
+1 & -1 \\
-1 & +1
\end{bmatrix}
\begin{bmatrix}
\cos \theta & \sin \theta & 0 & 0 \\
0 & 0 & \cos \theta & \sin \theta
\end{bmatrix}
\begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_3 \\
\delta_4
\end{bmatrix} =
\begin{bmatrix}
\Delta X \\
\Delta Y \\
\Delta Z \\
\Delta W
\end{bmatrix}
\]

and using the relation in (7):

Global force vector for element i:

Global stiffness matrix:

Local stiffness:

Rotation matrix:

Global stiffness of element i:

Global displacement vector for element i:
\[ K_i = T_i^T K_i T_i \]

\[ = \left( \frac{EA}{l} \right)_i \begin{bmatrix}
\cos^2 \theta & \cos \theta \sin \theta & -\cos^2 \theta & -\cos \theta \sin \theta \\
\cos \theta \sin \theta & \sin^2 \theta & -\cos \theta \sin \theta & -\sin^2 \theta \\
-\cos^2 \theta & -\sin^2 \theta & \cos^4 \theta & \sin^2 \theta \cos \theta \\
-\cos \theta \sin \theta & -\sin \theta \cos \theta & \sin \theta \cos \theta & \sin^2 \theta 
\end{bmatrix} \]

(9) for a truss structure, then we have to add the stiffness contribution at all elements and nodal points and assemble the global stiffness matrix as follows:

\[ K = \frac{EA}{l} \begin{bmatrix}
+1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & +1 & 0 \\
0 & 0 & 0 & 0 
\end{bmatrix} \]

\[ K = \frac{EA}{4l} \begin{bmatrix}
1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 \\
-1 & -1 & 1 & 1 
\end{bmatrix} \]

\[ \left( \frac{EA}{l} \right)_i \begin{bmatrix}
\frac{4+\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} \\\n\frac{\sqrt{2}}{4} & \frac{4+\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} \\
0 & 0 & 0 & 0 & 0 \\
-\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & \frac{\sqrt{2}}{4} \\\n-\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 & \frac{\sqrt{2}}{4} 
\end{bmatrix} \begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_3 \\
\delta_4 \\
\delta_5 
\end{bmatrix} = \begin{bmatrix}
R_A^x \\
R_A^y \\
R_B^x \\
R_B^y 
\end{bmatrix} \]
\[
\begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}
\begin{bmatrix}
\delta_u \\
\delta_K
\end{bmatrix}
= 
\begin{bmatrix}
F_k \\
F_u
\end{bmatrix}
\]

where \( \delta_u \) is unknown displacements, \( \delta_K \) is known displacements, \( F_k \) is known forces and \( F_u \) is unknown forces.

\[\begin{align*}
K_{11} \cdot \delta_u + K_{12} \cdot \delta_K &= F_k \\
\rightarrow \quad \delta_u &= K_{11}^{-1} \left( F_k - K_{12} \cdot \delta_K \right) \\
K_{21} \delta_u + K_{22} \cdot \delta_K &= F_u
\end{align*}\]

Calculating the unknown displacements.

Calculating support forces.

(10) Therefore, we can apply the above equations to our problem: \([\delta_K = 0]\)

\[
\frac{EA}{l} \begin{bmatrix}
1 + \frac{\sqrt{2}}{4} \\
\frac{\sqrt{2}}{4}
\end{bmatrix}
\begin{bmatrix}
\delta_3 \\
\delta_4
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
-P
\end{bmatrix}
\]

\[\rightarrow \quad \begin{bmatrix}
\delta_3 \\
\delta_4
\end{bmatrix}
= \frac{P l}{EA} \begin{bmatrix}
t + 1 \\
-3.83
\end{bmatrix}\]
11. We subsequently calculate the support forces, as follows:

\[
\begin{bmatrix}
-1 & 0 \\
0 & \frac{\sqrt{2}}{4} \\
0 & \frac{\sqrt{2}}{4} \\
\frac{-\sqrt{2}}{4} & \frac{-\sqrt{2}}{4}
\end{bmatrix}
\begin{bmatrix}
\delta_3 \\
\delta_4
\end{bmatrix}
= 
\begin{bmatrix}
R^x_A \\
R^x_C \\
R^y_A \\
R^y_C
\end{bmatrix}
= 
\begin{bmatrix}
-P \\
0 \\
+P \\
+P
\end{bmatrix}
\]

12. The last step involves calculating the internal force in each member:

\[
F_i = \left(\frac{EA}{l}\right)_i 
\begin{bmatrix}
-\cos\theta & -\sin\theta & \cos\theta & \sin\theta
\end{bmatrix}
\begin{bmatrix}
\delta_m \\
\delta_n \\
\delta_x \\
\delta_y
\end{bmatrix}
\]

where \( m \) and \( n \) belong to the nodes that on either sides of element \( i \).

\[\Theta = 0\]

\[
F_1 = \left(\frac{EA}{l}\right)_1 
\begin{bmatrix}
-1 & 0 & +1 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{Pl}{EA} \\
-3.83P/E_A \\
0 \\
0
\end{bmatrix}
= +P
\]

\[
F_2 = \left(\frac{EA}{l\sqrt{2}}\right)_2 
\begin{bmatrix}
-\frac{-\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2}
\end{bmatrix}
\begin{bmatrix}
\frac{Pl}{EA} \\
-3.83P/E_A \\
0 \\
0
\end{bmatrix}
= -P\sqrt{2}
\]

\[\Theta = -\frac{\pi}{4}\]